Dynamic Unravelling

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Abstract

This paper studies price and liquidity dynamics in the presence of costly short-selling when uninformed traders have limited willingness-to-pay to trade securities. In this setting, unravelling and Bayesian social learning interact to produce a novel mechanism, *dynamic unravelling*: unravelling that generates signals that lead to future unravelling. Applying the theory, I show how dynamic unravelling explains low-volume crashes: falls in the prices of securities on low or declining trading volume. In this context, short-selling restrictions can make low volume crashes more likely by intensifying dynamic unravelling but liquidity injections have the opposite effect.

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1 Introduction

Two important stylized facts in the empirical literature on market microstructure are that: (i) the degree of adverse selection matters for both the difference between buy and sell prices (the bid-ask spread) and trading volume (see *e.g.,* Huang and Stoll, 1997; Neal and Wheatley, 1998) and (ii) unusually low volume is predictive of future low returns (see *e.g.,* Gervais, Kaniel, and Mingelgrin, 2001; Akbas, 2016).

The classical approach for understanding these phenomena follows the dynamic market microstructure tradition of Diamond and Verrecchia (1987).[1] First, asymmetric information gives rise to adverse selection and a bid-ask spread. Second, the presence of short-selling costs makes lower volume informative that the value is low because informed traders who know that the value is low do not trade as it is too costly to short the security.[2]

However, this previous work assumes for analytical tractability that uninformed traders have infinite willingness-to-pay to transact. This is descriptively undesirable as there is a natural trade-off between the benefits of trading – perhaps derived from a need for liquidity, portfolio rebalancing, or irrational expectations – and the cost of so doing. Moreover, this assumption makes the counterfactual prediction (as per fact (i)) that trading volume is unchanged when spreads are wider (Huang and Stoll, 1997).

In this paper, I therefore relax this assumption and instead suppose that uninformed traders have a stochastic and finite willingness-to-pay to either buy or sell a security. The resulting joint dynamics for asset prices and liquidity reveal a novel mechanism, which I label *dynamic unravelling* (see Figure 1): increases in adverse selection not only drive unravelling today, but also generate signals that beget unravelling tomorrow. Concretely, when trading volume is low, the price of the security falls owing to the presence of short-selling costs. When such a fall in price increases uncertainty, this triggers an increase in bid-ask spreads. This rise in spreads leads uninformed traders with low idiosyncratic values from trading to drop out of the market, which statically increases adverse selection and triggers unravelling, as in Akerlof's (1970) market for lemons. Critically, uninformed traders dropping out of the market further decreases volume in the market, causing the mechanism to propagate into the future via a spiral of dynamically self-fulfilling unravelling.

As an application of the theory, I argue that dynamic unravelling provides an explanation for low volume crashes: large falls in the prices of financial securities on low or declining trading volume. I further study how common policy interventions – short-selling prohibitions and liquidity injections – affect the likelihood of low volume crashes.

[1] Which itself builds upon the seminal analysis of Glosten and Milgrom (1985).

[2] Akbas (2016) shows empirically that low volume is *more* predictive of future low returns when short-selling is more restricted, supporting this mechanism.

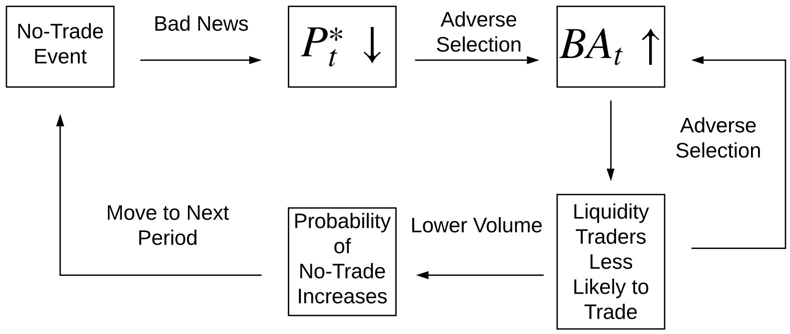

Figure 1: Flowchart of the Dynamic Unravelling Mechanism

The Dynamic Unravelling Mechanism. I build a dynamic model of trade in a financial market for a single security with informational asymmetries and costly short-selling. Risk-neutral traders come to the market each period and can either be informed of the value of the security or uninformed with idiosyncratic willingness-to-pay (noise traders). Upon arrival, traders submit price-quantity offers (limit orders) to market makers, who have a book size specifying the quantity of the security that they can transact. As traders are potentially privately informed about the value of the security, as in Glosten and Milgrom (1985), market makers face a winner's curse as demand is adversely selected, which they take into account in constructing their reservation prices. Imperfect information on the part of traders regarding market maker book capacity generates on-path rationing of trader demand and periods in which no-trade events occur.

Because short-selling is costly, no-trade events are themselves informative of the value of the financial security. In particular, an observation of no-trade can be due to an informationally neutral event such as a trader attempting to transact too many units of the security or a noise trader finding trade too costly, because their limited willingness-to-pay is insufficient to cross the bid-ask spread. However, as in Diamond and Verrecchia (1987), an observation of no-trade can also be due to the following informationally negative event: an informed trader wanting to short sell the security but finding so doing too costly. Thus, no-trade events are bad news regarding the fundamental value of the financial security and therefore give rise to a fall in price.

Static unravelling is triggered by such a no-trade event. In particular, when a no-trade event occurs in period $t-1$, the posterior belief of market makers that the value of the security is high falls, causing a fall in price $P_{t-1}^* > P_t^*$. Because uncertainty regarding the value of the security is maximized at the most uninformative belief, whenever P_{t-1}^* is sufficiently high, a fall in price causes an increase in uncertainty on the part of market makers.[3] This

[3]This is an outcome of the fact that the value of the security is assumed to be binary. More generally,

increase in uncertainty raises the degree of adverse selection faced by market makers and thus increases the bid-ask spread BA_t: the difference between the price market makers charge following a buy order (the ask price) and the price following a sell order (the bid price). This directly increases the cost to traders of transacting in the security and causes uninformed traders with insufficiently strong idiosyncratic demand to drop out of the market, creating a static feedback loop of rising bid-ask spreads, falling uninformed trader participation and increasing adverse selection, as in Akerlof's market for lemons (Akerlof, 1970).

The combination of unravelling and Bayesian social learning generates dynamic unravelling. Because static unravelling induces uninformed traders to drop out of the market, the likelihood of a no-trade event is larger in period t following a no-trade event in period $t-1$ than a no-trade event in period $t-1$ itself. As a result, a no-trade event in period $t-1$ begets static unravelling, which increases the chance in period t that another no-trade event occurs, which begets yet further static unravelling in period $t+1$, and so on. The effect of static unravelling increasing the likelihood of a signal that gives rise to yet more unravelling in future periods is what I label dynamic unravelling (see Figure 1) and is the primary conceptual contribution of the paper.

Application: Low Volume Crashes and Policy. Beyond its theoretical interest, dynamic unravelling generates long strings of no-trade events on which the price falls – thereby generating the auto-correlation in volumes and low volume crashes we observe in the data. Moreover, dynamic unravelling arises merely as a result of the interaction of short-selling constraints, asymmetric information, and learning – undoubtedly present features of financial markets. In Section 6, I discuss empirical support for the underlying mechanism and its predictions.

Having developed a theory that explains low volume crashes, I perform comparative statics and dynamics in the model to understand what role policy can have in alleviating the likelihood of a low volume crash. I show that higher short-selling costs can amplify the dynamic unravelling mechanism and give rise to a higher chance of a low volume crash. This is of practical interest as many practitioners advocate for short-selling prohibitions to arrest crashes.[4] While this analysis by no means implies that this is incorrect as it does not speak to all ways in which a crash can occur, it does suggest a novel trade-off that such policies may lead to the unintended consequence of increasing market risk of low volume crashes.

for dynamic unravelling to be operative, I require that bad news increases uncertainty.

[4]Most prominently, during the 2008 Financial Crisis, the US Securities and Exchange Commission (SEC) banned short-selling of 799 financial companies to prevent further price declines. SEC chairman Christopher Cox justified this by saying: "The commission is committed to using every weapon in its arsenal to combat market manipulation that threatens investors and capital markets ... The emergency order temporarily banning short selling of financial stocks will restore equilibrium to markets."

Conversely, liquidity injections reduce the chance of low volume crashes by increasing the chance of trade and arresting the dynamic unravelling mechanism.

Related Literature. This paper lies at the intersection of literatures on adverse selection and social learning in financial markets. Most relatedly, Eisfeldt (2004), Malherbe (2014), Chiu and Koeppl (2016), and Kurlat (2018) develop theories of liquidity dynamics in financial markets where asymmetric information can lead to adverse selection and a breakdown in trade.[5] Liquidity dynamics occur in my model through the margin of endogenous participation of noise traders, and the endogenous negative signal regarding asset quality generated by an absence of trade. Thus, relative to this literature and the seminal analysis of Diamond and Verrecchia (1987), I highlight a novel dynamic mechanism where a rise in adverse selection not only gives rise to static unravelling and an interruption in learning but also causes the market to generate price signals that give rise to more adverse selection in the future, generating declining trading volume, liquidity and prices in tandem.

The model features several innovations relative to Diamond and Verrecchia (1987). In particular, my analysis features price-quantity offers in a dynamic model with heterogeneous willingness-to-pay to trade, market maker illiquidity, and short-selling costs. Economically, this endogenizes the absence of trade as an outcome of one of three reasons: (i) insufficient willingness-to-pay to cross the bid-ask spread, (ii) insufficient market maker liquidity, and (iii) insufficient willingness to pay the short-selling costs. The first reason is the essential feature that generates trading volume that is elastic to the bid-ask spread. This is necessary to match the empirical evidence on the elasticity of volume to bid-ask spreads (as per Huang and Stoll, 1997; Neal and Wheatley, 1998) and is the core element that underlies the dynamic unravelling mechanism. The second and third reasons endogenize no-trade probabilities (which are taken as exogenous by Diamond and Verrecchia, 1987) as an outcome of optimizing behavior in a trading game. This yields economic dividends as it allows me to study counterfactuals for price dynamics. For example, I study how the probability of low-volume crashes responds to market maker liquidity.

On the technical side, my analysis features a signaling game with a double-continuum of

[5] In this context, Kurlat (2018) shows how a lack of liquidity leads to a breakdown in trade, which prevents learning, while Chiu and Koeppl (2016) focus on optimal policy in the presence of a related mechanism. Eisfeldt (2004) shows how the underlying productivity of the economy translates into liquidity. Malherbe (2014) studies a three period model in which agents hoarding cash increases adverse selection and leads to a breakdown in trade. Less directly related papers study how social learning depends on the level of economic activity in various contexts. More specifically, in Fajgelbaum, Schaal, and Taschereau-Dumouchel (2017) a lack of activity generates a higher dispersion of signals, not a negative signal as happens in my analysis. Moreover, in Straub and Ulbricht (2018), while uncertainty is self-reinforcing, this stems from a pledgeability constraint rather than adverse selection. Further papers that feature similar ideas include (Myerson and Satterthwaite, 1983; Kyle, 1985; Levin, 2001; Veldkamp, 2005; Van Nieuwerburgh and Veldkamp, 2006; Routledge and Zin, 2009; Attar, Mariotti, and Salani, 2011; Daley and Green, 2012).

states—one for each price-quantity combination—in a dynamic setting. However, I show that there exists a pooling equilibrium in which only the sign of trader demand is informative of valuations, rendering the price offer and the magnitude of the quantity offer informationally neutral.[6] This yields well-defined notions of bid and ask prices and reduces the dynamics of the model to a degree that is manageable by ensuring that from any state, there are only three states to which the model can transition, *i.e.,* those that follow a sell, a buy, or a no-trade event.

The dynamic unravelling mechanism moreover differs from that proposed by Dang, Gorton, and Holmström (2015), where debt (and other) securities become more information sensitive upon bad news. Rather than stemming from the static fact that bad news makes the price more sensitive, dynamic unravelling arises from an intertemporal feedback loop between rising adverse selection, bad news, and the uncertainty that bad news generates.

In its application to understanding low volume crashes, this paper relates to the literature on both liquidity and market crashes. The most related such papers are Daley and Green (2016) and Dow and Han (2018). Daley and Green (2016) consider a model where forward-looking agents have a demand for future liquidity and private information over the quality of their assets. When the market is pessimistic there is a high degree of adverse selection and a breakdown in trade until sufficiently good or bad news arrives. As a result, liquidity dries up and liquidity premia fall, causing a fall in prices. Dow and Han (2018) show how when informed traders are liquidity constrained, prices become less informative, adverse selection rises, the supply of high-quality assets falls and prices fall. In contrast to both of these papers, in my model lack of trade is itself informative and causes a self-reinforcing spiral of adverse selection, falling trade and prices. My model therefore generates the very large falls in prices required to match the data even in the absence of an evolving underlying state.[7]

Outline. The rest of the paper proceeds as follows. Section 2 introduces the model. Section 3 characterizes the static properties of equilibrium. Section 4 characterizes equilibrium price dynamics and demonstrates dynamic unravelling. Section 5 shows how dynamic unravelling causes low volume crashes and how liquidity injections and short-selling constraints affect the likelihood of low volume crashes. Section 6 discusses empirical evidence and concludes.

[6] I moreover provide a natural sufficient condition – noise traders' valuations admit a density – such that there are no separating or semi-separating equilibria, and all equilibria are of the type that I study.

[7] My paper also relates to the large literature on market crashes (Gennotte and Leland, 1990; Romer, 1993; Kodres and Pritsker, 2002; Barlevy and Veronesi, 2003; Hong and Stein, 2003; Yuan, 2005; Huang and Wang, 2008). As in Huang and Wang (2008), my model has the desirable feature that it generates low volume crashes but not low volume surges. Critically, these models of market crashes feature crashes that occur on high volume. Relative to these papers, my model differs along the margin that it generates the low volume crashes we observe in the data.

2 Model

2.1 Primitives

There is a finite time horizon T and time is discrete $t \in \{0, \ldots, T\}$. There is a single financial security with date T liquidation value $V \in \{0, 1\}$ that is drawn at date 0 with $\mathbb{P}[V = 1] = p \in (0, 1)$. In each period t, there are two agents: a market maker and a trader. Both enter the market at the beginning of period t and leave the market before period $t + 1$.

The timing of the game between the trader and market maker in period t is as follows. First, traders make limit orders $o_t = (p_t, x_t)$ specifying a price at which they'd be willing to trade p_t and a quantity x_t. Then, the market maker has book capacity $\bar{x}_t$, and can decide whether or not to accept the proposed offer. Trade occurs when (i) the trader's offer is accepted and (ii) the trader's proposed quantity is not in excess of the market maker's capacity. If either of these conditions fails to hold, then no-trade will be executed. All trades, but not offers, are public and the public history is therefore given by $h^t = \{(p_\tau, x_\tau)\}_{0 \leq \tau \leq t} \in \mathcal{H}^t$. Note, importantly, that if no trade is observed, then the history records an absence of trade with $(\emptyset, \emptyset)$ and the original offer is not observed. This simple trading protocol allows for tractable analysis. In Section 6.2, I discuss how my results are robust to considering alternative trading protocols.

I now describe in full detail how I model market makers and traders.

2.1.1 Market Makers

The market makers are risk-neutral and Bayesian. Hence, conditional on an offer o_t and a history h^{t-1}, they have reservation value given by:

$$P(h^{t-1}, o_t) = \mathbb{E}[V | h^{t-1}, o_t] \tag{1}$$

That is, they are willing to buy the security if $p_t \leq P(h^{t-1}, o_t)$ and sell the security if $p_t \geq P(h^{t-1}, o_t)$.

Market makers have finite capacity to absorb trading volume in the form of a book capacity $\bar{x}_t$. Concretely, the market maker can accept trades only if the demanded quantity is below their book capacity $|x_t| \leq \bar{x}_t$. The capacity $\bar{x}_t$ is drawn each period from a history-dependent distribution, which allows past trades, and therefore the current shadow price of the security, to affect the current capacity of market makers:

$$\bar{x}_t \sim G(\cdot | h^{t-1}) \tag{2}$$

This flexible specification can therefore capture, for example, both physical inventory limits and a mark-to-market value constraint. I assume that, for all t and h^{t-1}, $G(\cdot|h^{t-1})$ has full support on a closed interval and a continuous PDF $g(\cdot|h^{t-1})$ satisfying the technical conditions that (i) there exists an $x \in \text{Supp}G(\cdot|h^{t-1})$ such that $x > \frac{1-G(x|h^{t-1})}{g(x|h^{t-1})}$ and (ii) $2g(x|h^{t-1}) + xg'(x|h^{t-1}) > 0$ for all $x \in \text{Supp}G(\cdot|h^{t-1})$.[8]

This feature is present in my model for two reasons. First, it is realistic and allows for policy analysis of how liquidity interventions that increase market maker ability to absorb trading volume affect the likelihood of low volume crashes. Second, it generates on-path rationing of trader demand and thereby periods in which no trade occurs. However, as I clarify in Section 6.2, this exact specification is not necessary for the main results: it would suffice to assume that traders have unit demand and that there is an exogenous (potentially history-dependent) probability $\delta(h^{t-1})$ that trade fails to take place, perhaps because no trader arrives to the market (as in Diamond and Verrecchia, 1987).

2.1.2 Traders

Traders are informed of the value of V with probability π. With the complementary probability $1-\pi$, traders arriving after any history h^{t-1} value the security as:

$$\tilde{V}(h^{t-1}) = \mathbb{E}[V|h^{t-1}] + \eta \tag{3}$$

where η is drawn from CDF K with $0 < K(0) < K(1) < 1$.[9] Primitively, these traders are uninformed and have liquidity demands given by η for standard idiosyncratic reasons such as portfolio balancing or beliefs, but are rational and infer the common value of the security from all publicly available information. Importantly, noise traders have finite idiosyncratic demand instead of the standard assumption of infinite willingness-to-pay in Glosten and Milgrom (1985) and Diamond and Verrecchia (1987). Hence, the participation of noise traders in the market is endogenous: when spreads are wide, fewer noise traders are willing to transact. This will be crucial for the dynamic unravelling mechanism, as I later clarify.

There is an IID short-selling cost $c \sim J$ of selling each unit of the security. The interpretation as a short-selling cost and not merely a transaction cost is facilitated by the fact that c is stochastic. Concretely, agents possessing the security who would not face a short-selling cost are accommodated as agents that receive a draw of $c = 0$.

[8] This technical condition serves to ensure that the agent has an interior optimal demand policy.

[9] This assumption is made to ensure that some fraction of noise traders are always willing to trade.

Traders with valuation $\hat{V}$ therefore have Bernoulli utility given by:

$$U(p, x, \hat{V}) = x(\hat{V} - p + c\mathbb{I}[x < 0]) \tag{4}$$

Moreover, the probability that a trade is executed given an offer o_t, the history h^{t-1} and a conjectured reservation value for the market maker P_t is:

$$\mathbb{P}[\text{Trade}|o_t, h^{t-1}] = \begin{cases} \mathbb{P}[|x_t| \leq \bar{x}_t, p_t \geq P_t|o_t, h^{t-1}], \ x_t > 0, \\ \mathbb{P}[|x_t| \leq \bar{x}_t, p_t \leq P_t|o_t, h^{t-1}], \ x_t < 0. \end{cases} \tag{5}$$

The trader's problem is therefore:

$$\max_{(p_t, x_t)} \mathbb{P}[\text{Trade}|o_t, h^{t-1}]U(p_t, x_t, \hat{V}) \tag{6}$$

The solution to this problem is denoted by an offer function $o^*(h^{t-1}, \hat{V}) = (p^*(h^{t-1}, \hat{V}), x^*(h^{t-1}, \hat{V}))$. Any solution to the trader's problem will naturally trade-off the gains from transacting larger volumes with the risk of not being able to transact.

2.2 Equilibrium

An equilibrium in this context is a standard rational expectations equilibrium and consists of an offer function for traders that solves the trader's problem and a reservation value function that is given by the conditional expectation of the value of the security given all publicly available information.

Definition 1. *An equilibrium is a tuple (P, o) comprising the market maker's reservation value function P, and the trader's offer function o such that:*

1. *The offer function maximizes trader expected utility $o = o^*$*
2. *Market makers have reservation value satisfying $P(h^{t-1}, o_t) = \mathbb{E}[V|h^{t-1}, o_t]$*

where all expectations are taken with knowledge of (P, o).

3 Static Equilibrium Analysis

Before studying dynamics, I first study the trading game between traders and market makers.

3.1 A Simple Class of Equilibria

I establish the existence of pooling equilibria where reservation prices depend only on the sign of demand and neither its magnitude nor the offered price in the following result:

Theorem 1. *There exists an equilibrium such that the market maker's reservation price depends only on the history and the sign of the trader's demand, i.e.,*

$$P(h^{t-1}, x) = \begin{cases} A(h^{t-1}) = \mathbb{E}[V|x > 0, h^{t-1}], & x > 0, \\ N(h^{t-1}) = \mathbb{E}[V|x = 0, h^{t-1}], & x = 0, \\ B(h^{t-1}) = \mathbb{E}[V|x < 0, h^{t-1}], & x < 0. \end{cases} \tag{7}$$

Traders always set their price offer at the market maker's reservation price corresponding to the sign of their demand.

Proof. See Appendix A.1. □

The proof of this result proceeds in three steps. First, I guess and verify that traders' price offers are informationally neutral, as all traders simply offer the market maker their reservation price. Second, I show that, if market makers believe the size of trader demand is unrelated to whether or not a trader is informed, and traders know this, then the magnitude of traders' demand is unrelated to traders' informedness. Implicitly, this equilibrium is supported by the (correct) beliefs of traders that higher demand implies nothing about the valuation of the trader. The final step of the proof verifies that such an equilibrium exists via a standard fixed point argument.

Thus, there exist equilibria in which all that matters is if there is a buy, sell, or no-trade. This reduces the state-space of the model from a potential double continuum (all offers) to three states, and enables tractable analysis of the dynamical system of prices and beliefs.

Given that there could also exist separating equilibria, why should we focus on such equilibria? In addition to their tractability for the dynamic analysis, I argue these equilibria are a natural benchmark to study for two reasons. The first is empirical. In a pooling equilibrium, there are endogenously well-defined bid and ask prices. In any separating equilibrium, bid and ask prices depend on order size. However, empirical evidence from equity markets is supportive of a (very) weak relationship between order sizes and bid-ask spreads. For example, in a classic analysis of 150 NYSE-traded stocks, Lin, Sanger, and Booth (1995) find that going from a bottom quartile trade size to a top one percentile trade size increases the quoted spread by only 6% (See Table 2 in Lin, Sanger, and Booth, 1995). The second is theoretical. I show in Appendix Lemma 2 that there exist no pure strategy fully (or semi-separating) equilibria in the natural case where the distribution of noise trader valuations admits a density. Therefore, it is not only empirically reasonable, but also best posed theoretically, to focus on pooling equilibria.

Which of my assumptions are essential for the existence of equilibria in which only the sign of the trade matters? First, the public history assumption ensures that traders offer prices equal to market maker reservation values. Relaxing this assumption would cause traders to be endogenously risk averse about not meeting reservation values and would cause them to shade their offered prices. This would introduce a valuation dependence of price offers on trader information and would cause market makers to perform non-trivial inference based on price offers. Second, the risk-neutrality assumption of traders is what allows the separability that ensures there is only information in the sign of demand. Relaxing this assumption would again introduce a dependence of the magnitude of quantity offers on trader information as traders would shade their quantities in a way the depends on their valuation. Again, this would lead market makers to condition on the magnitude of trader demand. Both of these forces are potentially interesting, but lie outside of the present analysis.

3.2 Prices in Equilibrium

For the remainder of the analysis, I therefore study equilibria of the kind derived in Theorem 1. The only relevant information in trader offers o_t is given by events in the set:

$$\{B = \mathbb{I}[x_t > 0], N = \mathbb{I}[x_t = 0], S = \mathbb{I}[x_t < 0]\} \tag{8}$$

where B corresponds to buy, N corresponds to no-trade and S corresponds to sell. Thus, we have well-defined bid prices $B(h^{t-1}) = \mathbb{E}[V|S, h^{t-1}]$, ask prices $A(h^{t-1}) = \mathbb{E}[V|B, h^{t-1}]$, and shadow prices following no-trade $N(h^{t-1}) = E[V|N, h^{t-1}]$. The shadow price following no-trade is never observed but serves to index the common posterior of the fundamental value in the absence of trade.

Toward studying dynamics, I first study the static properties of prices. To this end, denote the equilibrium *rationing probability* that a trader demands in excess of the market maker's capacity as:

$$\delta(h^{t-1}) = \mathbb{E}\left[G\left(|x^*(h^{t-1}, \hat{V})|\ |h^{t-1}\right)|h^{t-1}\right] \tag{9}$$

Important static properties of prices are given in the following result.

Proposition 1. *Conditional on a history h^{t-1}:*

1. *The bid price is always lower than yesterday's shadow price $P^*_{t-1} \equiv \mathbb{E}[V|h^{t-1}]$ and the ask price always exceeds yesterday's shadow price:*

$$B_t \leq P^*_{t-1} \leq A_t \tag{10}$$

2. *Bid and ask prices do not depend on the distribution of market maker capacity G*
3. *The no-trade shadow price always lies below the previous shadow price:*

$$N_t \leq P^*_{t-1} \tag{11}$$

4. *The shadow price following no trade N_t is increasing in $\delta(h^{t-1})$.*

Proof. See Appendix A.3. □

The first point establishes that, owing to adverse selection, traders selling is bad news and traders buying is good news. The second point shows that bid and ask prices are, conditional on a history, independent of the distribution of market maker capacity G. The reason for this is simply that as informed and uninformed traders are symmetrically rationed, there is no effect of liquidity on the pool of traders conditional on trade occurring.

The third point establishes that no-trade events are bad news and lies at the heart of the later analysis. The reason is intuitive: an observation of no trade can be caused by informed or uninformed agents being symmetrically rationed, an uninformed trader having insufficient willingness-to-pay to transact, or an informed trader who wishes to short sell finding short-selling too costly. Thus, the states giving rise to no-trade are either neutral or negative and so no trade represents bad news. The final point shows that, when the rationing probability goes up, it is more likely that an observation of no-trade is caused by a neutral state of the world in which a trader is simply unable to transact, rather than an informed agent finding short-selling too costly. Hence, as $\delta(h^{t-1})$ rises, no-trade represents less bad news.

A final interesting feature of the model is that bid and no-trade prices cannot generally be ordered. In particular, no-trade events can be worse news about the underlying security than an outright sell. This may seem counter-intuitive, but stems simply from the fact that no-trade can be a more precise signal of low asset quality in a world with high short-selling costs (as few informed traders short sell) and (absolutely) high idiosyncratic valuations for noise traders (as few uninformed traders do not trade).

4 Dynamic Unravelling

I now study the short-run dynamics of prices and establish conditions under which dynamic unravelling occurs.

4.1 Characterization of Price Dynamics

I begin by characterizing the dynamical system for prices. Applying Bayes' rule, the evolution of prices can be expressed in odds-ratio form. Owing to Theorem 1, all that remains to characterize the stochastic process for prices is the mapping from the primitives of the model to $\mathbb{P}[x|V=v,h^{t-1}]$ for $x \in \{B,S,N\}$ and $v \in \{0,1\}$. Doing this yields Proposition 2.

Proposition 2. *Prices evolve according to:*

$$\frac{P_t^*(x,h^{t-1})}{1-P_t^*(x,h^{t-1})} = \frac{P_{t-1}^*}{1-P_{t-1}^*}\frac{\mathbb{P}[x|V=1,h^{t-1}]}{\mathbb{P}[x|V=0,h^{t-1}]} \tag{12}$$

for $x \in \{B,S,N\}$ *and where:*

$$\begin{aligned}
\mathbb{P}[B|V=1,h^{t-1}] &= (1-\delta(h^{t-1}))\left[\pi + (1-\pi)(1-K(A(h^{t-1})-P_{t-1}^*))\right]\\
\mathbb{P}[B|V=0,h^{t-1}] &= (1-\delta(h^{t-1}))\left[(1-\pi)(1-K(A(h^{t-1})-P_{t-1}^*))\right]\\
\mathbb{P}[S|V=1,h^{t-1}] &= (1-\delta(h^{t-1}))\left[(1-\pi)\mathbb{E}_c[K(B(h^{t-1})-P_{t-1}^*-c)]\right]\\
\mathbb{P}[S|V=0,h^{t-1}] &= (1-\delta(h^{t-1}))\Big[\pi J(B(h^{t-1})) + (1-\pi)\mathbb{E}_c[K(B(h^{t-1})-P_{t-1}^*-c)]\Big]
\end{aligned} \tag{13}$$

with the no-trade probabilities given by the residuals and the unconditional probabilities of x *given by the law of iterated expectations applied to Equation 13.*

Proof. See Appendix A.4. □

As these expressions are somewhat involved, Figure 2 plots the first order difference equation for shadow prices, where I have chosen the parameter values to illustrate the full range of interesting phenomena present in the model. The Figure shows the new shadow price to which the system transitions for each of a buy, sell and no-trade event as a function of the previous shadow price, which indexes the common prior as to the value of the security. In conjunction with Proposition 1, one immediately sees that $B_t, N_t \leq P_{t-1}^* \leq A_t$ by comparing each of the lines to the 45 degree line.[10]

There are three further interesting properties to note. First, there is a region for high P_{t-1}^* such that no-trade events are informationally neutral and $N_t = P_{t-1}^*$. This stems

[10] Because the shadow price is a Martingale in the model, this also allows one to see the expected future prices in the model over all future time paths.

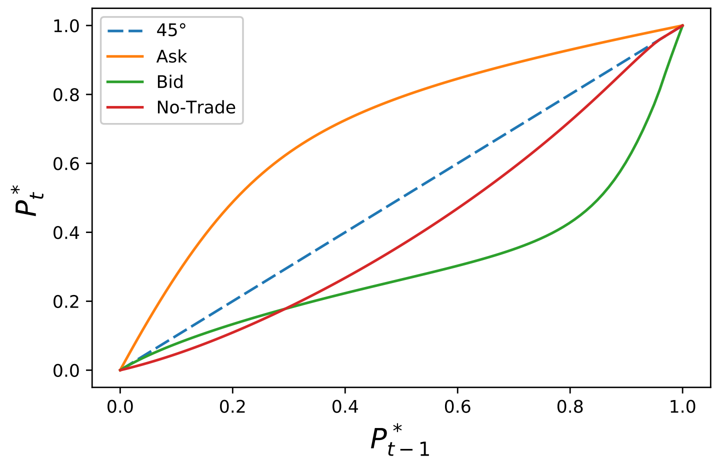

Figure 2: Illustration of the Price Transition Functions. Parameters: $p = 0.7, \pi = 0.5, \eta \sim U[-1, 1], c \sim U[0, 0.8], \bar{x} \sim N(6, 9)$.

from bounding the short-selling cost away from one so that for sufficiently high P_{t-1}^* all informed traders who know the value is low attempt to short sell. This indicates a potentially interesting state-dependence as to the informational content of no trade.

Second, there can exist a region for low P_{t-1}^* such that no-trade events provide a worse signal for the value of the security than outright sells $N_t < B_t$. As alluded to earlier, this stems from the fact that as P_{t-1}^* falls to zero, there are very few informed traders willing to pay the short-selling cost as the gains from shorting are small while, owing to (absolutely) large idiosyncratic valuations, uninformed traders remain willing to short sell. Consequently, no-trade becomes a more precise signal of low value than a sell.

Third, when market makers are pushed toward being uninformed (intermediate P_{t-1}^*) the size of the bid-ask spread increases. This stems from two forces. First, greater uninformedness increases the marginal impact of information. This follows from the standard assumption that the asset's value is binary, which implies that uncertainty is maximized when prices are intermediate. Thus, starting from a high price, falls in prices on bad news translate into increases in uncertainty, which manifest as a large bid-ask spread. The exact parametric form of a binary support is necessary for tractability in studying dynamics, but the key economic force behind dynamic unravelling appears robust in any situation where bad news increases uncertainty. Second, as noise traders have finite willingness-to-pay, their endogenous trading decisions generate static unravelling, which I now discuss.

4.2 Adverse Selection and Static Unravelling

The combination of adverse selection and endogenous noise trader participation manifest as static unravelling in the sense of Akerlof (1970). As we have argued, when the market maker has an intermediate belief, they are most uncertain and spreads are therefore large. Owing to endogenous noise trader participation, there is a further force in the model that reinforces this basic statistical property: when bid-ask spreads widen, uninformed traders are more likely to have an idiosyncratic valuation that leaves their value between the bid and ask, causing them to drop out of the market. To see this in the model, recall that a noise trader's valuation is given by:

$$\hat{V}(h^{t-1}) = \mathbb{E}[V|h^{t-1}] + \eta = P^*_{t-1} + \eta \tag{14}$$

Thus, a noise trader's choice of whether to buy, sell or not trade given bid and ask prices $B(h^{t-1})$ and $A(h^{t-1})$ can be simply described as: buy if $\eta \geq A(h^{t-1}) - P^*_{t-1}$; sell if $\eta \leq B(h^{t-1}) - P^*_{t-1} - c$; and do not trade otherwise. As a result, the fraction of noise traders who drop out of the market $\Delta(h^{t-1})$ is given by:

$$\Delta(h^{t-1}) = K(A(h^{t-1}) - P^*_{t-1}) - \mathbb{E}_J[K(B(h^{t-1}) - P^*_{t-1} - c)] \tag{15}$$

which is clearly increasing in the ask price and decreasing in the bid price, and therefore in the bid-ask spread all else equal. For example, in the uniform valuations case $\eta \sim U[\underline{\eta}, \overline{\eta}]$ when there are no shorting costs, $\mathbb{P}_J[c = 0] = 1$, we have that the fraction of noise traders who drop out of the market is linear in the bid-ask spread:

$$\Delta(h^{t-1}) = \frac{BA(h^{t-1})}{\underline{\eta} + \overline{\eta}} \tag{16}$$

Thus, larger bid-ask spreads increase the fraction of informed traders in the pool of traders who transact, making demand more adversely selected and causing wider bid-ask spreads. This further reduces noise trader participation, increases adverse selection and spreads, and mirrors the classic unravelling mechanism of Akerlof (1970).

The above discussion is formalized in Lemma 1, which simply states that in the absence of endogenous noise trader participation, the ask function is concave and the bid function is convex in the shadow price. Moreover, when the ask is concave and the bid convex, there is a maximal bid-ask spread that occurs at a value of the previous period's shadow price P', below which the bid-ask spread in increasing and above which the bid-ask spread is decreasing.

Lemma 1. *The following properties of bid prices, ask prices and the bid-ask spread hold:*

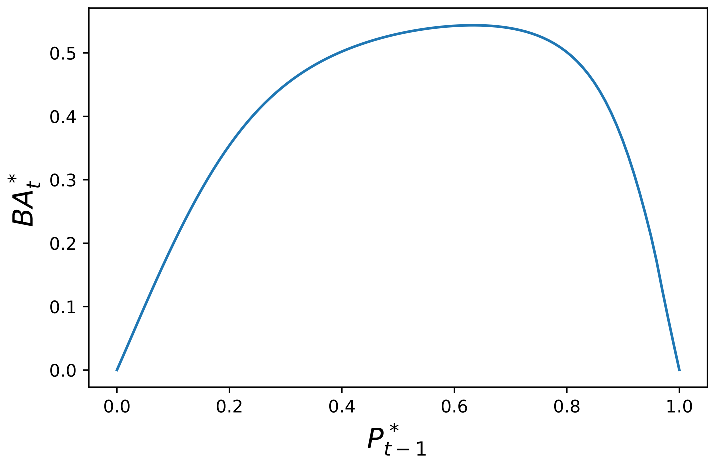

Figure 3: Illustration of the Bid-Ask Spread Being Monotone Either Side of its Maximum. Parameters: $p = 0.7, \pi = 0.5, \eta \sim U[-1, 1], c \sim U[0, 0.8], \bar{x} \sim N(6, 9)$.

1. *In the absence of endogenous noise trader participation,*[11] *the ask function A is concave in P_{t-1}^* and the bid function B is convex in P_{t-1}^*.*
2. *If A is a concave function of P_{t-1}^* and the bid function B is a convex function of P_{t-1}^*, then there exists a $P'(h^{t-1})$ such that $BA_t \equiv A_t - B_t$ is decreasing in P_{t-1} for all $P_{t-1} > P'(h^{t-1})$ and increasing in P_{t-1} for all $P_{t-1} < P'(h^{t-1})$.*

Proof. See Appendix A.5. □

More generally in the presence of noise trader participation, a full characterization of concavity or convexity of the bid and ask functions, and therefore the presence of unravelling a la Akerlof (1970), is challenging owing to arbitrary non-linearity of the fixed point equations for bid and ask prices. However, I provide a sufficient condition for concave ask and convex bid prices in the following footnote,[12] and verify the property numerically in the case of uniform valuations (see Figure 3).

[11] Formally, take $K = K'$, where K' is defined in Equation 21.

[12] To verify that the ask price is concave (an analogous argument can be applied to show the bid price is convex), by Proposition 2 it is sufficient to show that the solution to the fixed point equation:

$$\xi(A(P), P) = \frac{A(P)}{1 - A(P)} - \frac{P}{1 - P}\left(\frac{\pi}{1 - \pi}\left(1 - K(A(P) - P)\right)^{-1} + 1\right) = 0 \tag{17}$$

has a negative second derivative in P. By twice applying the implicit function theorem, we thereby obtain the following sufficient condition:

$$A''(P) = -\frac{1}{\xi_1}\left[\xi_{11}\left(\frac{\xi_2}{\xi_1}\right)^2 - (\xi_{12} + \xi_{21})\frac{\xi_2}{\xi_1} + \xi_{22}\right] < 0 \tag{18}$$

for all P, where the partial derivatives of ξ are evaluated at the solution to the fixed point equation.

4.3 From Static Unravelling to Dynamic Unravelling

In traditional analyses of unravelling, the above is where the mechanism stops. However, in the present model, note that not only has unravelling increased the bid-ask spread today, it has increased the probability of another no trade event as fewer noise traders trade. As a result, the model features endogenous auto-correlation in no-trade events: in the region of the state space with declining bid-ask spreads and operative static unravelling, failure to trade today increases the probability that we fail to trade tomorrow. This claim is formalized in Proposition 3.

Proposition 3. *Fix $\delta(h^{t-1}) = \delta(h^{t-2})$ and suppose that A is concave in P^*_{t-1} and B is convex in P^*_{t-1}.*[13] *Following an observation of no-trade at history h^{t-2}, the probability of no-trade at history $h^{t-1} = (h^{t-2}, N)$ is larger if $P(h^{t-1}) > P'(h^{t-1})$. That is:*

$$\mathbb{P}[N|h^{t-1}] > \mathbb{P}[N|h^{t-2}] \tag{19}$$

Proof. See Appendix A.6. □

I now explain how this endogenous auto-correlation establishes dynamic unravelling. I have established that when a no-trade event occurs for a sufficiently high initial price, the following will occur: agents will be forced back towards being uninformed by the bad news caused by no trade, which will cause a fall in prices and a rise in bid-ask spreads. This causes noise traders to drop out of the market (as more are unwilling to cross the now larger bid-ask spread) which leads to a static feedback loop (static unravelling) of increasing bid-ask spreads and noise traders dropping out (as per Equation 15). In turn, the fact that noise traders drop out of the market increases the probability that another no-trade is realized. Moreover, following another no-trade event, this mechanism begins yet again. As a result, the model features a dynamic feedback loop where no-trade events beget unravelling in the next period and unravelling begets no-trade events – the dynamic unravelling mechanism depicted in the introduction as Figure 1.

To make the claim that dynamic unravelling generates endogenous auto-correlation precise, I can consider a within-model counterfactual where I "turn off" endogenous noise trader participation and thereby dynamic unravelling. Dynamic unravelling should make sequences of no-trade events more likely.[14] However, if all noise traders had sufficiently large willingness-to-pay for the security that they would never drop out of the market, then the dynamic

[13] A sufficient condition for $\delta(h^{t-1}) = \delta(h^{t-2})$ is that G does not depend on the history.

[14] An alternative counterfactual is to simply assume that no-trade is informationally neutral, *i.e,* $\mathbb{P}[N|(N, h^{t-2})] = \mathbb{P}[N|h^{t-2}]$. Proposition 3 then immediately implies that a sequence of k no-trade events is more likely under dynamic unravelling.

unravelling mechanism is not operative and we should find that sequences of no-trades are less likely than in the model with dynamic unravelling present.

Formally, define $S(k)$ as the event that there is a sequence of $k \in \mathbb{N}$ sequential no-trade events starting at some given history. The probability of any event $S(k)$ can be expressed by applying Bayes' rule iteratively as:

$$\mathbb{P}[S(k)|h^{t-1}] = \prod_{i=1}^{k} \mathbb{P}[N|N^{(k-i)}, h^{t-1}] \tag{20}$$

where $\mathbb{P}[N|N^{(k-i)}, h^{t-1}]$ refers to the probability of a no-trade event conditional on a history following h^{t-1} with $k-i$ no trade events and no other events.

I now show how the dynamic unravelling mechanism increases this probability relative to the counterfactual model without the force. Formally, take any valuation distribution for noise traders K and define $\gamma(K) = \mathbb{P}[\eta \geq 0] = 1 - K(0)$ as the fraction of noise traders that have idiosyncratic demand to buy the security. Now construct a new probability distribution of noise trader valuations K' such that:

$$\mathbb{P}_{K'}[\eta = x] = \begin{cases} \gamma(K), & x = \alpha, \\ 1 - \gamma(K), & x = -\alpha, \\ 0, & \text{otherwise.} \end{cases} \tag{21}$$

for some constant $\alpha > 1 + \sup\{\text{Supp}J\}$. This new distribution of valuations has mass $\gamma(K)$ of noise traders who will always buy and mass $1-\gamma(K)$ of noise traders who will always sell. See that under K' noise traders will never drop out of the market owing to either short-selling costs or being caught between the bid-ask spread. Moreover, K' preserves the measure of traders who wish to buy and sell under K. In this sense, comparing the model with K and K' isolates the effect of endogenous noise trader participation and the dynamic unravelling mechanism.

Proposition 4 establishes formally that the likelihood of any sequence of no trade events is greater in the model featuring dynamic unravelling.

Proposition 4. *Fix a P^*_{t-1} and $\delta(h^{t-1})$. For any $k \in \mathbb{N}$, the probability of a sequence of k no-trade events $S(k)$ is larger under K than under K':*

$$\mathbb{P}_K[S(k)|h^{t-1}] \geq \mathbb{P}_{K'}[S(k)|h^{t-1}] \tag{22}$$

Proof. See Appendix A.7. □

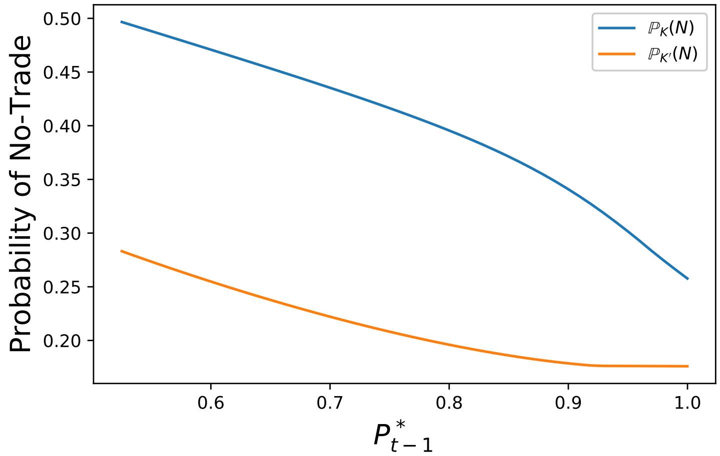

Figure 4: No-Trade Probabilities with and Without Dynamic Unravelling. Parameters (blue line): $p = 0.7, \pi = 0.5, \eta \sim U[-1, 1], c \sim U[0, 0.8], \bar{x} \sim N(6, 9)$. Parameters (orange line): same except $\eta \sim K'$.

To exemplify the role played by dynamic unravelling, Figure 4 plots the no-trade probabilities with and without endogenous noise trader participation over the region of the state space with bid-ask spreads that rise when the price falls ($P^*_{t-1} \geq P'$). Over this region, each no-trade event lowers the shadow price P^*_t (see Figure 2) and increases bid-ask spreads (see Figure 3), generating dynamic unravelling. That the no-trade probability is increasing under K is guaranteed by Proposition 3. Proposition 4 implies that the probability under K (blue line) must lie above the probability under K' (orange line). In this example, one can moreover observe in the model with endogenous noise trader participation that when the price falls and spreads rise, the no-trade probability rises by more than the model without noise trader participation.[15]

Having formally established the dynamic unravelling mechanism, in the next sections I explain how dynamic unravelling gives rise to low volume crashes, examine empirical support, and discuss its policy implications. However, before I do this, there are two additional features of the dynamic unravelling mechanism to note. First, whenever the price is sufficiently higher than $P'(h^{t-1})$ (the point of maximized bid-ask spreads) the mechanism strengthens as it progresses. Figure 4 shows this diagrammatically: starting from a sufficiently high price, when a no-trade event occurs and the price falls, the no-trade probability continues to increase. Second, the implications of dynamic unravelling are directional. This is because no-trade events always impart either negative or neutral information, and so the model does not generate any increase in prices while no-trade occurs. Thus, in the region of the state

[15]The rising no-trade probability in the model without noise trader participation is driven by the fact that agents are more likely unwilling to pay short-selling costs when the price is lower.

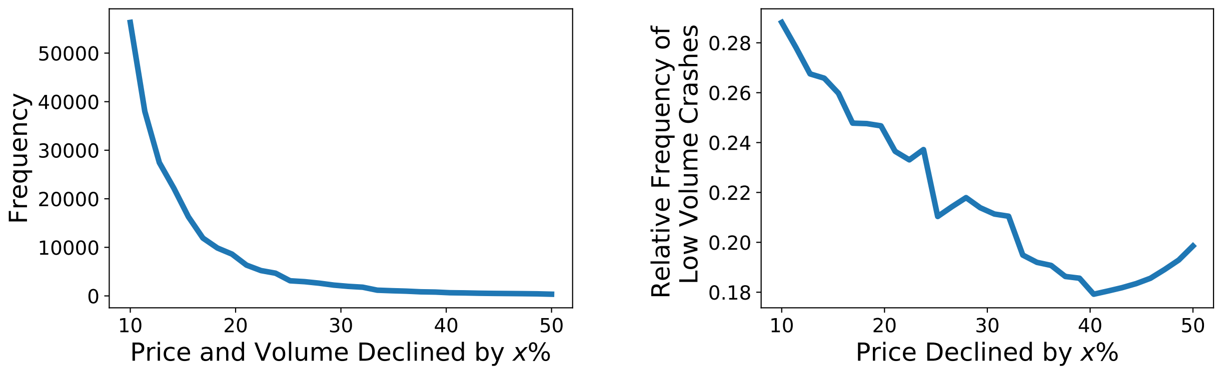

Figure 5: Left: Frequency of Low Volume Crashes, Right: Relative Frequency of Crashes in Price by $x\%$ While Volume Fell by 10% or More to Total Crashes in Price by $x\%$. Data: Day-to-day Price and Volume Changes from the Universe of CRSP Data from 1963-2019.

space with increasing bid-ask spreads, the analogous dynamic feedback loop between adverse selection and no-trade does not simply operate in reverse.

5 Application: Low Volume Crashes and Policy

Beyond being of simply theoretical interest, dynamic unravelling provides a rationale for the occurrence of low volume crashes, or large falls in the price of assets on low trading volume. These events are ubiquitous for both financial securities (Campbell, Grossman, and Wang, 1993) and physical assets such as housing (Clayton, Miller, and Peng, 2010; DeFusco, Nathanson, and Zwick, 2017). To this point, using the universe of Center for Research in Security Prices (CRSP) data on equities, Figure 5 documents (i) the number of days in which price and volume both declined by $x\%$ for any equity and (ii) the relative number of day-to-day declines in price by $x\%$ during which volume declined by at least 10%. One sees immediately a striking fact: there are a very large number of crashes that occur on declining volume, both absolutely and relative to total crashes. In particular, around one quarter of all crashes occur on declining volume.[16] Understanding the causes of such low volume crashes is therefore critical for both a complete understanding of market crashes and the design of financial policy.

5.1 Dynamic Unravelling as a Cause of Low Volume Crashes

To establish formally how dynamic unravelling cause low volume crashes, I first define a stylized model counterpart of a low volume crash as k sequential no-trade events such that

[16]As Figure 5 shows, low volume crashes do become slightly less represented the larger a crash becomes, but always constitute at least 18% of all crashes on which the price declines by anything up to 50%.

the price falls each period. Formally, a *low volume crash* is an event where for some t_0 and k, $P_{t_0+i} > P_{t_0+j}$ for all $i, j \in \{0, 1, \ldots, k\}$ and $j > i$, and there is no trade for all $t \in \{t_0, \ldots, t_0 + k - 1\}$. The following immediate corollary of Proposition 4 crystallizes the sense in which dynamic unravelling makes low volume crashes likely relative to models that do not have endogenous noise trader participation such as Diamond and Verrecchia (1987):

Corollary 1. *A low volume crash of any length $k \in \mathbb{N}$ is more likely in the model featuring dynamic unravelling K than in the model without dynamic unravelling K'.*

Intuitively, dynamic unravelling makes low volume crashes more likely as it increases the likelihood of long strings of no-trades on which the price falls.

5.2 Policies to Avert Dynamic Unravelling and Low Volume Crashes

I now analyze the possible implications of dynamic unravelling for financial policy by studying the impact of common policies on the likelihood of low volume crashes.

First, short-selling restrictions are often proposed as a means to arrest crashes. However, in light of dynamic unravelling, there is good reason to suppose that they might increase the risk of low volume crashes. In particular, by increasing short-selling costs, the negative information content of trade is heightened, increasing the scope for a low volume crash.

To see this possibility formally, suppose for analytical tractability that the distribution of noise trader-valuations is uniform $\eta \sim U[\underline{\eta}, \overline{\eta}]$ for $\underline{\eta} < 0$ and $\overline{\eta} > 1$. This simplifies the fraction of noise traders who drop out of the market to a linear function of bid-ask spreads. Moreover, suppose that short-selling costs are either prohibitively expensive with probability λ or do not exist with probability $1 - \lambda$. The policymaker can increase (or decrease) short-selling costs to some level λ'. Under these assumptions, I show that an increase in short-selling costs may increase the likelihood of a low volume crash:

Proposition 5. *Suppose that $A(h^{t-1})$ and $B(h^{t-1})$ are respectively concave and convex functions in P^*_{t-1}. If short selling costs rise to $\lambda' > \lambda$:*

$$\mathbb{P}_{\lambda'}[S(k)|h^{t-1}] > \mathbb{P}_{\lambda}[S(k)|h^{t-1}] \tag{23}$$

for all $k \leq \bar{k}$ where $\bar{k} \geq 1$ whenever π is sufficiently large.

Proof. See Appendix A.8. □

Intuitively, increasing short-selling costs reduces the quantity of both informed and uninformed trade and endogenously worsens inference conditional on no trade (when π is sufficiently large). Thus, no trade is both more likely and causes a greater reduction in prices, worsening the dynamic unravelling loop. This result is potentially important given the popularity of short-selling prohibitions as a means of arresting crashes as it suggests that they may increase the market risk of low volume crashes.

If short-selling restrictions are ineffective in arresting low volume crashes, what tools might aid policymakers? Liquidity injections that support market makers' ability to transact are ideally suited to this endeavor as they encourage trade and thereby arrest the dynamic unravelling mechanism. Formally, suppose that the policymaker can intervene by changing $\bar{x}_t$. A smaller $\bar{x}_t$ can be interpreted as a position limit and a looser $\bar{x}_t$ as a liquidity injection. By changing position limits, the planner manipulates the rationing probability $\delta(h^{t-1})$, changing the chance of a no-trade event, and thereby a low volume crash:

Proposition 6. *Suppose that noise trader valuations are given by K'. Consider a position limit intervention whereby the planner increases $\bar{x}_t$ by $\xi > 0$. For any $k \in \mathbb{N}$, we have that:*

$$\mathbb{P}_{\xi}[S(k)|h^{t-1}] < \mathbb{P}_0[S(k)|h^{t-1}] \tag{24}$$

Proof. See Appendix A.9. □

This proposition establishes the intuitive result that liquidity injections by a policymaker can reduce the chance of low volume crashes. The assumption I make on noise traders' valuations serves to ensure that the direct effect of reducing the chance of no-trade is not outweighed by a somewhat pathological indirect effect of so much worsening inference conditional on no-trade that the policy backfires.

So far my analysis of policy has taken reducing the likelihood of crashes as a given policy objective. However, the formalism of the model admits a natural welfare metric whereby the planner's objectives can be endogenized (Proposition 8 in Online Appendix B.2) . This allows an analysis of optimal policy, both static (Proposition 9 in Online Appendix B.2) and dynamic (Section B.2.3), from the perspective of the welfare of market participants. Price discovery and preventing crashes emerge as natural desiderata when one uses the implied welfare metric, providing a microfoundation for this qualitative analysis of policy.

In Online Appendix B.1, I further derive the impact of short-selling costs and position limits on price discovery. That is, I derive comparative statics on the speed of learning (the expected time for the price to become sufficiently close to zero or one) and show that more liquidity speeds learning and higher short-selling costs slow learning (Proposition 7 in Online Appendix B.1). Thus, higher position limits both improve informational efficiency and blunt the likelihood of low volume crashes. Higher short-selling costs, however, both reduce informational efficiency and increase the likelihood of low volume crashes.

6 Discussion and Conclusion

In this section, I discuss the empirical plausibility of the dynamic unravelling mechanism and the role of the key assumptions underlying the model, and then conclude.

6.1 The Empirical Plausibility of Dynamic Unravelling

The plausibility of the dynamic unravelling mechanism relies on two premises: no trade is bad news because of short-selling constraints; and low volume in one period leads to low volume in the following period. I argue that there is evidence for both of these premises in the empirical literature on market microstructure.

First, the mechanism relies on the static loop where, owing to short-selling constraints, no trade is bad news. This relies crucially on short-selling costs being significant in the markets under consideration and some *ceteris paribus* relationship between short-selling costs and volume. To this first point, Davolio (2002) using data on every US equity security over the 18-month period from April 2000 to September 2001 from a large financial institution, notes that around 9% of stocks have shorting fees that average 4.3% per annum and about 10% of stocks are never sold short. This provides strong evidence of significant shorting costs in equity markets. To the second point, Reed (2002) uses data on the entire daily loan portfolio from an anonymous, large US securities lender from 1998 to 1999 and finds that both trading volume falls and prices become less informative when short-selling is constrained. Moreover, on announcement days, the fraction of long run price reaction realized on the day of the announcement is smaller when short-selling is constrained – providing evidence of slower information transmission in the presence of greater short-selling constraints. Thus, there is direct evidence of substantial short-selling costs in the markets under consideration and therefore scope for the mechanism I outline to have quantitative bite.

Furthermore, there is strong direct evidence of no-trade representing bad news in equity markets and that this is caused to a large degree by short-selling costs. Gervais, Kaniel, and Mingelgrin (2001) show that unusually low trading volume, over the period of a day or week, predicts lower returns. Akbas (2016) tests whether this is driven by the informational content of low volume. In particular, they show that unusually low trading volume before earnings announcements predicts more unfavorable earnings surprises. Moreover, they show that the relationship between low volume and unfavorable earnings surprises is stronger for stocks with higher short-selling cost as measured by short interest.[17] This provides direct evidence for the key model mechanism that low volume is bad news and that short-selling constraints

[17] They also show, using content analysis of firm-specific news, that low trading volume predicts negative sentiment in news reports. Yet further, high volume does not predict positive sentiment.

exacerbate the negative content of low volume. Moreover, these predictions distinguish my analysis from that of herding models (such as Froot, Scharfstein, and Stein, 1992; Park and Sabourian, 2011) that do not feature such a mechanism.

Second, the mechanism relies on a dynamic feedback loop where low volume in one period leads to low volume in the following period. A necessary condition for this is that we observe auto-correlation in trading volumes and that low volume is predictive of future low volume. Campbell, Grossman, and Wang (1993), Chordia and Swaminathan (2000) and a large subsequent literature document these predictions in the universe of CRSP data on US equities. Yet more specifically, the model predicts that low volume in one period should lead to lower uninformed volume in the following period and a higher proportion of informed traders. Easley, Engle, O'Hara, and Wu (2008) provide suggestive evidence of these effects using a structural autoregressive model with time-varying flows of informed and uninformed agents that they estimate using a sample of 16 actively traded stocks over a period of 15 years of transaction data. First, they find that less trade in one period generates both less uninformed and informed trade.[18] Second, they find that there is less arrival of uninformed traders if there is more past informed trade. As more past informed trade should translate into greater adverse selection and wider spreads, this provides suggestive evidence of uninformed traders leaving the market when spreads widen, which is the key step in the dynamic unravelling mechanism.

6.2 Discussion of Assumptions

I finally inspect how the conclusions I have drawn are robust to the main assumptions.

First, I have assumed that the asset's value is binary. As I have discussed, this has the implication that uncertainty is maximal for intermediate prices. While the assumption of binary values was necessary for tractable dynamic analysis, all propositions in the static analysis are in fact proved under the richer condition that V has CDF F on $[\underline{V}, \overline{V}]$ with continuous PDF f. Speculatively, one should expect dynamic unravelling to occur in any situation where bad news increases uncertainty. Moreover, there are good empirical reasons to suppose that this is the case given the strong correlation between bad news and uncertainty in financial markets (as per the well-documented strong and negative correlation between the S&P500 and the VIX) and the real economy (see *e.g.,* Bloom, Floetotto, Jaimovich, Saporta-Eksten, and Terry, 2018).

Second, I have assumed a specific trading protocol where an order is either filled in its

[18] While the model as written does not predict lower informed trade, a small modification whereby informed traders also have some degree of idiosyncratic valuation would introduce this prediction. I do not study this in the formal analysis for analytical tractability.

entirety or not filled at all. While this is true of limit order systems implemented on many exchanges and is a sensible main specification, it is important to emphasize that my results are robust to changes in this trading protocol. In particular, so long as the model features some chance that trade fails to take place for reasons that have nothing to do with the fundamental value of the security, then all of my analysis will carry through. For example, if I eliminated capacity constraints entirely (while retaining willingness-to-pay heterogeneity and short-selling costs) and instead assumed as in Diamond and Verrecchia (1987) that there is a probability κ that no trader arrives to the market in any given period, then the predictions of my model with a constant $\delta = \kappa$ do not change: dynamic unravelling obtains and all propositions up to this point (except Proposition 6 on position limits) hold.

Third, I have assumed that short-selling costs are exogenous. This assumption can be relaxed to allow the distribution of short-selling costs to depend on the current market price. In this case, we must index the CDF of short-selling costs and write J_P. When the cost depends on the actual price, all results hold as stated, with J_P replacing J. Intuitively, this does not upset the dynamic unravelling results as the market price does not change along a sequence of no-trades (as no transactions are actually happening).[19]

Fourth, I have assumed that market maker liquidity constraints are symmetric. It is simple to relax this assumption and instead assume that the market maker has a capacity to sell x^+, drawn from G^+, and a capacity to buy x^-, drawn from G^-. In this case, trade happens if $x \in [x^-, x^+]$. It is simple to verify that bid and ask prices remain independent of both G^+ and G^-. The no-trade probabilities must be adjusted to account for both G^+ and G^-, but suitable modifications of all propositions hold as stated.[20]

Finally, given I have modelled traders as acting myopically, it is interesting to consider how my conclusions may be altered by allowing traders to time their trades so as to minimize market impact as in Kyle (1985). Naturally, when bid-ask spreads are wide and few noise traders are present in the market, a forward looking informed buyer will have incentive to delay trade yet further. While beyond the formal analysis, one anticipates that this would have two implications. On the one hand, the probability of no-trade would be yet higher, strengthening the dynamic unravelling mechanism. On the other hand, the degree of adverse selection in bad states would be tempered if informed traders care more about timing their trades than uninformed traders. It would be an interesting, albeit challenging, avenue of

[19] If instead, J depended on the previous shadow price, then the dynamic unravelling results (Propositions 3 and 4) hold so long as short-selling costs are more severe when the asset is undergoing a crash, *i.e.,* $J_{P^*} \succeq_{FOSD} J_{P^{*'}}$ for all $P^* \leq P^{*'}$.

[20] More formally, we must define δ^+ and δ^- with G^+ and G^- replacing G in the definition of δ (Equation 9). Propositions 3 and 4 then hold when the hypothesis that $\delta(h^{t-1}) = \delta(h^{t-2})$ is replaced with the hypotheses that $\delta^+(h^{t-1}) = \delta^+(h^{t-2})$ and $\delta^-(h^{t-1}) = \delta^-(h^{t-2})$.

future research to investigate which of these effects dominates and whether these forward-looking considerations dampen or amplify the core logic of dynamic unravelling.

6.3 Conclusion

In this paper, I study the implications of rational uninformed traders in the presence of short-selling costs for the dynamics of asset prices and liquidity. The key theoretical contribution of the paper is to show how dynamic unravelling can occur: observations of no-trade not only induce unravelling in the sense of Akerlof (1970) but also make it more likely that another observation of no-trade occurs, increasing the likelihood of future unravelling. As a primary application, I show how dynamic unravelling provides an explanation for low volume crashes. In this context, the implications for policy are novel. Short-selling constraints can amplify the dynamic unravelling mechanism and make low volume crashes more likely. This suggests a new trade-off for policymakers considering the use of short-selling constraints to arrest market crashes. I show that liquidity injections, however, may have the opposite effect.

Appendices

A Omitted Proofs of Main Results

A.1 Proof of Theorem 1

Proof. I prove this result for the richer case that V has CDF F on $[\underline{V}, \overline{V}]$ with continuous PDF f. In the binary case considered in the main text, all integrals over V can be replaced with the conditional probability that $V = 1$ and all results carry. The proof first shows that there are equilibria where the reservation value doesn't depend upon the offered price. I then show by a guess and verify argument that there are equilibria in which only the sign of demand matters for the reservation price. Finally, I show that such an equilibrium exists.

Step 1: Price Strategies Are Uninformative. I first show that it is weakly dominant for traders to choose price-offering strategies that are equal to market maker reservation values evaluated at the trader's offer:

$$p(h^{t-1}, \hat{V}) = P(h^{t-1}, o(h^{t-1}, \hat{V})) \tag{25}$$

for all time periods t, histories h^{t-1}, and valuations $\hat{V}$. As h^{t-1} is public and o is the trader's choice variable, $P(h^{t-1}, o)$ is measurable with respect to the information of the trader. Suppose the agent offers $p > P(h^{t-1}, o)$. This is dominated by $p = P(h^{t-1}, o)$. Moreover, offering $p < P(h^{t-1}, o)$ is dominated by $p = P(h^{t-1}, o)$. $p = P(h^{t-1}, o)$ therefore dominates all potential offers and $P(h^{t-1}, o) = p(h^{t-1}, \hat{V})$ is therefore optimal.

From this, it follows that market maker reservation values do not depend on the prices submitted by traders in such an equilibrium:

$$P(h^{t-1}, o) = P(h^{t-1}, x) \tag{26}$$

To see this, from the previous step, we have that $P(h^{t-1}, x^*(h^{t-1}, \hat{V}), p(h^{t-1}, \hat{V})) = p(h^{t-1}, \hat{V})$. Fixing $x^*(h^{t-1}, \hat{V}) = x$, we then have that $P(h^{t-1}, x, p(h^{t-1}, \hat{V})) = p(h^{t-1}, \hat{V})$. From the equation for market maker reservation values we have:

$$P(h^{t-1}, x, p(h^{t-1}, \hat{V})) = \mathbb{E}[V|h^{t-1}, x, p(h^{t-1}, \hat{V})] \tag{27}$$

We now guess and verify the result. Suppose that all agents regardless of $\hat{V}$ submit the same price conditional on x. It follows that:

$$\mathbb{E}[V|h^{t-1}, x, p(h^{t-1}, \hat{V})] = \mathbb{E}[V|h^{t-1}, x] \tag{28}$$

Thus:

$$P(h^{t-1}, x, p(h^{t-1}, \hat{V})) = \mathbb{E}[V|h^{t-1}, x] \tag{29}$$

which is not a function of p. Thus $P(h^{t-1}, o) = P(h^{t-1}, x)$. Moreover, given that $P(h^{t-1}, o) = P(h^{t-1}, x)$, we have that $P(h^{t-1}, x) = p$ under the optimal policy, verifying the conjecture.

Step 2: Trade Magnitudes Are Uninformative. Given this, the trader's problem is given by:

$$\max_{x_t} (1 - G(|x_t||h^{t-1})) \times (\hat{V} - P(h^{t-1}, x_t) + c\mathbb{I}[x_t < 0]) \times x_t \tag{30}$$

We now guess and verify that:

$$P(h^{t-1}, x) = \begin{cases} A(h^{t-1}) = \mathbb{E}[V|x > 0, h^{t-1}], & x > 0, \\ N(h^{t-1}) = \mathbb{E}[V|x = 0, h^{t-1}], & x = 0, \\ B(h^{t-1}) = \mathbb{E}[V|x < 0, h^{t-1}], & x < 0. \end{cases} \tag{31}$$

Given this price function, conditional on knowing the optimal sign of x_t, to determine the optimal magnitude of x_t, the trader necessarily solves the following FOC:

$$(1 - G(|x_t||h^{t-1})) \times \text{sign}(x_t) - g(|x_t||h^{t-1})x_t = 0 \tag{32}$$

which has no dependence on $\hat{V}$. Thus, conditional on $\text{sign}(x_t)$, x_t indeed conveys no information about $\hat{V}$. Formally:

$$P(h^{t-1}, x_t) = \mathbb{E}[V|h^{t-1}, x_t] = \mathbb{E}[V|h^{t-1}, \text{sign}(x_t)] = P(h^{t-1}, \text{sign}(x_t)) \tag{33}$$

This verifies the conjecture.

Step 3: Trading Strategies. To complete a description of equilibrium, we need to determine the sign of trader demand. See that this is pinned down by four cases:

1. $\hat{V} \geq A(h^{t-1}) \implies x_t > 0$
2. $\hat{V} \in (B(h^{t-1}), A(h^{t-1})) \implies x_t = 0$
3. $\hat{V} \in (B(h^{t-1}) - c, B(h^{t-1})] \implies x_t = 0$
4. $\hat{V} \leq B(h^{t-1}) - c \implies x_t < 0$

Hence, the trader's optimal demand function is given by:

$$x(h^{t-1}, \hat{V}) = \begin{cases} \frac{1 - G(x(h^{t-1}, \hat{V})|h^{t-1})}{g(x(h^{t-1}, \hat{V})|h^{t-1})}, & \hat{V} \geq A(h^{t-1}), \\ 0, & \hat{V} \in (B(h^{t-1}), A(h^{t-1})), \\ 0, & \hat{V} \in (B(h^{t-1}) - c, B(h^{t-1})] \\ \frac{G(x(h^{t-1}, \hat{V})|h^{t-1}) - 1}{g(x(h^{t-1}, \hat{V})|h^{t-1})}, & \hat{V} \leq B(h^{t-1}) - c. \end{cases} \tag{34}$$

To check that such an equilibrium exists, it remains to show two things. First, that there is an $x_t(h^{t-1}, \hat{V})$ that solves Equation 34 for any $P(h^{t-1}, x)$ of the specified form. Second, given $x_t(h^{t-1}, \hat{V})$, there is a solution to Equation 31.

To prove the first of these points, see that we need to guarantee that there exists an x' such that:

$$0 = 1 - G(x') - x'g(x') \tag{35}$$

Define $l(x) = 1 - G(x) - xg(x)$. See that $l(0) = 1 > 0$ and by assumption that there exists an x such that $x > \frac{1-G(x)}{g(x)}$ there exists an x such that $l(x) < 0$. Moreover, as G and g are continuous and full support, l is continuous. Thus, by the intermediate value theorem there exists an x' such that $l(x') = 0$. We moreover need such a point to be optimal. This is guaranteed by the condition $2g(x|h^{t-1}) + xg'(x|h^{t-1}) > 0$ for all $x \in \text{Supp}G(\cdot|h^{t-1})$, as this implies that the trader's problem is globally concave.

Step 4: Existence of Ask Price. To prove the second point, we need to prove that there exists a solution to Equation 31. To this end, see that the ask function is given by:

$$
\begin{aligned}
A(h^{t-1}) &= \mathbb{E}[V|x>0,h^{t-1}] \\
&= \int_{\underline{V}}^{\overline{V}} \tilde{v} f(\tilde{v}|x>0,h^{t-1}) d\tilde{v} \\
&= \int_{\underline{V}}^{\overline{V}} \tilde{v} \frac{\mathbb{P}[x>0|\tilde{v},h^{t-1}] f(\tilde{v}|h^{t-1})}{\mathbb{P}[x>0|h^{t-1}]} d\tilde{v} \\
&= \frac{\int_{\underline{V}}^{\overline{V}} \tilde{v} \mathbb{P}[x>0|\tilde{v},h^{t-1}] f(\tilde{v}|h^{t-1}) d\tilde{v}}{\mathbb{P}[x>0|h^{t-1}]}
\end{aligned} \tag{36}
$$

Moreover, see that the probability of a buy is given by the following step function:

$$
\mathbb{P}[x>0|V,h^{t-1}] = \begin{cases} (1-\delta(h^{t-1}))\left[\pi + (1-\pi)(1-K(A(h^{t-1})-P^*_{t-1}))\right], \ V \geq A(h^{t-1}) \\ (1-\delta(h^{t-1}))(1-\pi)(1-K(A(h^{t-1})-P^*_{t-1})), \ V < A(h^{t-1}) \end{cases} \tag{37}
$$

We therefore obtain the following fixed point equation for the ask price:

$$
\begin{aligned}
A(h^{t-1}) &= \frac{\pi \int_{A(h^{t-1})}^{\overline{V}} \tilde{v} f(\tilde{v}|h^{t-1}) d\tilde{v} + (1-\pi)(1-K(A(h^{t-1})-P^*_{t-1}))P^*_{t-1}}{(1-F(A(h^{t-1})|h^{t-1}))\pi + (1-\pi)(1-K(A(h^{t-1})-P^*_{t-1}))} \\
&\equiv \alpha(A(h^{t-1});h^{t-1})
\end{aligned} \tag{38}
$$

Under the technical assumption that $0 < K(\underline{V}) < K(\overline{V}) < 1$, we have that $\alpha(h^{t-1}) : [\underline{V},\overline{V}] \to [\underline{V},\overline{V}]$. Hence, the domain and co-domain of $\alpha(h^{t-1})$ are equal and compact. Thus, existence of a fixed point will follow by Brouwer's fixed point theorem if $\alpha(h^{t-1})$ is continuous. To this end, consider a sequence $\{A_n\}_{n\in\mathbb{N}}$ such that $A_n \in [\underline{V},\overline{V}]$ for all $n \in \mathbb{N}$. First take $A_n \to A \in (\underline{V},\overline{V})$. Consider now $\alpha(A_n;h^{t-1})$. Note that $\alpha(A_n;h^{t-1})$ is in fact differentiable in A_n for $A_n \in (\underline{V},\overline{V})$. Thus continuity on the interior is ensured. To prove continuity at the boundaries, now take a sequence $A_n \to \overline{V}$. The following is immediate:

$$
\lim_{n\to\infty} \alpha(A_n;h^{t-1}) = \alpha(\overline{V};h^{t-1}) = P^*_{t-1} \tag{39}
$$

Finally, take a sequence $A_n \to \underline{V}$. The following is, once again, immediate:

$$
\lim_{n\to\infty} \alpha(A_n;h^{t-1}) = \alpha(\underline{V};h^{t-1}) = P^*_{t-1} \tag{40}
$$

Hence, $\alpha(h^{t-1})$ is continuous and Brouwer's fixed point theorem ensures that there exists the required fixed point.

Step 5: Existence of Bid Price. Having shown existence of the ask, we now follow a similar technique to prove existence of the bid. To this end, see that the bid function is given by:

$$\begin{aligned} B(h^{t-1}) &= \mathbb{E}[V|x<0,h^{t-1}] \\ &= \int_{\underline{c}}^{\overline{c}}\int_{\underline{V}}^{\overline{V}} \tilde{v} f(\tilde{v}|x<0,c,h^{t-1}) d\tilde{v} dJ(c) \\ &= \int_{\underline{c}}^{\overline{c}}\int_{\underline{V}}^{\overline{V}} \tilde{v}\frac{\mathbb{P}[x<0|\tilde{v},c,h^{t-1}]f(\tilde{v}|h^{t-1})}{\mathbb{P}[x<0|c,h^{t-1}]} d\tilde{v} dJ(c) \\ &= \int_{\underline{c}}^{\overline{c}} \frac{\int_{\underline{V}}^{\overline{V}} \tilde{v}\mathbb{P}[x<0|\tilde{v},c,h^{t-1}]f(\tilde{v}|h^{t-1})d\tilde{v}}{\mathbb{P}[x<0|c,h^{t-1}]} dJ(c) \end{aligned} \tag{41}$$

The probability of a sell is given by the step function:

$$\mathbb{P}[x<0|c,h^{t-1}] = \begin{cases} (1-\delta(h^{t-1}))(1-\pi)K(B(h^{t-1})-c-P^*_{t-1}), & V \geq B(h^{t-1})-c \\ (1-\delta(h^{t-1}))\left[\pi + (1-\pi)K(B(h^{t-1})-c-P^*_{t-1})\right], & V < B(h^{t-1})-c \end{cases} \tag{42}$$

Thus, the fixed point equation for the bid can be expressed as:

$$\begin{aligned} B(h^{t-1}) &= \int_{\underline{c}}^{\overline{c}} \frac{\pi\int_{\underline{V}}^{\min\{B(h^{t-1})-c,\overline{V}\}} \tilde{v} f(\tilde{v}|h^{t-1})d\tilde{v} + (1-\pi)K(B(h^{t-1})-c-P^*_{t-1})P^*_{t-1}}{\pi F(B(h^{t-1})-c|h^{t-1}) + (1-\pi)K(B(h^{t-1})-c-P^*_{t-1})} dJ(c) \\ &= \beta(B(h^{t-1});h^{t-1}) \end{aligned} \tag{43}$$

Under the technical assumption that $0 < K(\underline{V}) < K(\overline{V}) < 1$, we have that $\beta(h^{t-1}) : [\underline{V},\overline{V}] \to [\underline{V},\overline{V}]$. Hence, the domain and co-domain of $\beta(h^{t-1})$ are equal and compact. Thus, existence of a fixed point will follow by Brouwer's fixed point theorem if $\beta(h^{t-1})$ is continuous. To this end, consider a sequence $\{B_n\}_{n\in\mathbb{N}}$ such that $B_n \in [\underline{V},\overline{V}]$ for all $n \in \mathbb{N}$. First take $B_n \to B \in (\underline{V},\overline{V})$. Consider now $\beta(B_n;h^{t-1})$. Note that $\beta(B_n;h^{t-1})$ is in fact differentiable in B_n for $B_n \in (\underline{V},\overline{V})$ except at possibly a measure 0 set. Thus continuity on the interior is ensured. To prove continuity at the boundaries, now take a sequence $B_n \to \overline{V}$. The following is immediate:

$$\lim_{n\to\infty} \beta(B_n;h^{t-1}) = \beta(\overline{V};h^{t-1}) = P^*_{t-1} \tag{44}$$

Finally, take a sequence $B_n \to \underline{V}$. The following is, once again, immediate:

$$\lim_{n\to\infty} \beta(B_n; h^{t-1}) = \beta(\underline{V}; h^{t-1}) = P^*_{t-1} \tag{45}$$

Hence, $\beta(h^{t-1})$ is continuous and Brouwer's fixed point theorem ensures that there exists the required fixed point.

Finally, we need to show existence of a no trade price. This is immediate given a fixed point for $A(h^{t-1})$ and $B(h^{t-1})$ as:

$$N(h^{t-1}) = \mathbb{E}[V|x = 0, h^{t-1}] \tag{46}$$

and the probability that $x = 0$ is pinned down by primitives as well as $A(h^{t-1})$ and $B(h^{t-1})$. This completes the proof. □

A.2 Statement and Proof of Lemma 2

Lemma 2. *If K admits a density, then there exist no fully separating equilibria and no semi-separating equilibria.*

Proof. Without loss of generality consider time $t = 0$. Suppose there exists a pure strategy fully separating equilibrium with demands $X : \mathbb{R} \to \mathbb{R}$ where $X(\hat{V})$ is the demanded quantity of valuation type $\hat{V}$. As the equilibrium is fully separating, X is invertible. We know the informed trader with $V = 1$ will demand some $X(1)$ at $t = 0$. Moreover, all noise traders with valuations $\hat{V}$ will demand some $X(\hat{V})$ which equals $X(1)$ if and only if $\hat{V} = 1$. As the informed trader has demand $X(1)$ with probability $p\pi$ and there is a density of $\hat{V}$, the trade $X(1)$ is perfectly revealing that the state is $V = 1$ and so $P(X(1)) = 1$ and the informed type has zero expected utility. However, for any $X(\hat{V}) \neq X(1)$, as no informed traders submit this order, there is no information in such trades and $P(X(\hat{V})) = p$ for all $\hat{V} \neq 1$. Thus, informed type $V = 1$ can imitate a type $\hat{V} = 1 - \varepsilon$ and obtain strictly positive expected utility for any $\varepsilon < 1$. This is a contradiction, so there cannot exist any pure strategy separating equilibria.

Suppose now that there exists a semi-separating equilibrium. That is, agents play a mixed strategy $\sigma : \mathbb{R} \to \Delta(\mathbb{R})$, where $\sigma(\hat{V}) \in \Delta(\mathbb{R})$ is the distribution of trades played by valuation type $\hat{V}$ at time $t = 0$. For all $x \in \text{Supp}\sigma(\hat{V})$, for indifference on the support of the conjectured mixed strategy we require that:

$$k(\hat{V}) = (1 - G(x))(\hat{V} - P(x) + c\mathbb{I}[x < 0])x \tag{47}$$

for some $k(\hat{V})$. This implies that the price satisfies:

$$P(x) = \hat{V} + c\mathbb{I}[x < 0] - \frac{k(\hat{V})}{(1 - G(x))x} \tag{48}$$

for all $\hat{V}$ and all $x \in \text{Supp}\sigma(\hat{V})$. This immediately implies that the supports of the mixed strategies for different valuation types must be disjoint as otherwise $P(x)$ depends on $\hat{V}$. But then, for any $x \in \text{Supp}\sigma(1)$, we have that $P(x) = 1$ as this is perfectly revealing that the value of the security is 1 as there is a density of noise trader valuations. But then we have that $P(x) = p$ for all $x \notin \text{Supp}\sigma(1), \text{Supp}\sigma(0)$. This implies that there is a unique strictly optimal $x^*(\hat{V})$ as we assumed that $2g(x) + xg'(x) > 0$. But then as before, type $\hat{V} = 1$ has a strict incentive to imitate any other type. This is a contradiction. Thus, there exist no semi-separating equilibria. □

A.3 Proof of Proposition 1

Proof. I prove this result for richer case that V has CDF F on $[\underline{V}, \overline{V}]$ with continuous PDF f. In the binary case, all integrals over V can be replaced with the conditional probability that $V = 1$ and all results carry. I prove the four parts of the proposition in a slightly different order of (ii), (i), (iii), (iv).

1. See that ask and bid prices are given by:

$$\begin{aligned} A(h^{t-1}) &= \mathbb{E}[V|x > 0, h^{t-1}] = \int_{\underline{V}}^{\overline{V}} \tilde{V} f(\tilde{V}|x > 0, h^{t-1}) d\tilde{V} \\ B(h^{t-1}) &= \mathbb{E}[V|x < 0, h^{t-1}] = \int_{\underline{V}}^{\overline{V}} \tilde{V} f(\tilde{V}|x < 0, h^{t-1}) d\tilde{V} \end{aligned} \tag{49}$$

   Moreover, applying Bayes' rule these conditional densities are given by:

$$\begin{aligned} f(V|x > 0, h^{t-1}) &= \frac{\mathbb{P}[x > 0|V, h^{t-1}] f(V|h^{t-1})}{\mathbb{P}[x > 0|h^{t-1}]} \\ f(V|x < 0, h^{t-1}) &= \frac{\mathbb{P}[x < 0|V, h^{t-1}] f(V|h^{t-1})}{\mathbb{P}[x < 0|h^{t-1}]} \end{aligned} \tag{50}$$

   To compute the denominators, we apply the law of total probability (doing so pointwise for each c and then averaging over c by the law of iterated expectations):

$$
\begin{aligned}
\mathbb{P}[x>0|h^{t-1}] &= \mathbb{P}[V \geq A(h^{t-1})|h^{t-1}]\mathbb{P}[x>0|V \geq A(h^{t-1}), h^{t-1}] \\
&\quad + \mathbb{P}[V < A(h^{t-1})|h^{t-1}]\mathbb{P}[x>0|V < A(h^{t-1}), h^{t-1}] \\
\mathbb{P}[x<0|h^{t-1}] &= \mathbb{E}_J\Big[\mathbb{P}[V \geq B(h^{t-1}) - c|c, h^{t-1}]\mathbb{P}[x<0|V \geq B(h^{t-1}) - c, c, h^{t-1}] \\
&\quad + \mathbb{P}[V < B(h^{t-1}) - c|c, h^{t-1}]\mathbb{P}[x<0|V < B(h^{t-1}) - c, c, h^{t-1}]\Big]
\end{aligned}
\tag{51}
$$

The leading part of each term here is simply given by $1 - F(y|h^{t-1})$ or $F(y|h^{t-1})$ for the obvious value of y to which that term pertains. The conditional parts can be computed:

$$
\begin{aligned}
\mathbb{P}[x>0|V \geq A(h^{t-1}), h^{t-1}] &= \pi(1-\delta(h^{t-1})) \\
&\quad + (1-\pi)(1-\delta(h^{t-1}))(1-K(A(h^{t-1}) - P^*_{t-1})) \\
\mathbb{P}[x>0|V < A(h^{t-1}), h^{t-1}] &= (1-\pi)(1-\delta(h^{t-1}))(1-K(A(h^{t-1}) - P^*_{t-1})) \\
\mathbb{P}[x<0|V \geq B(h^{t-1}) - c, c, h^{t-1}] &= (1-\pi)(1-\delta(h^{t-1}))K(B(h^{t-1}) - c - P^*_{t-1}) \\
\mathbb{P}[x<0|V \geq B(h^{t-1}) - c, c, h^{t-1}] &= \pi(1-\delta(h^{t-1})) \\
&\quad + (1-\pi)(1-\delta(h^{t-1}))K(B(h^{t-1}) - c - P^*_{t-1})
\end{aligned}
\tag{52}
$$

See that the factor $(1-\delta(h^{t-1}))$ is common to all terms of both numerators and denominators. Thus, the factor $(1-\delta(h^{t-1}))$ cancels and neither $A(h^{t-1})$ nor $B(h^{t-1})$ depend on $\delta(h^{t-1})$. As G enters the model only through $\delta(h^{t-1})$, this yields the result.

2. To prove that $B(h^{t-1}) \leq P^*_{t-1} \leq A(h^{t-1})$ we need to show:

$$
\int_{\underline{V}}^{\overline{V}} \tilde{V} f(\tilde{V}|x<0, h^{t-1}) d\tilde{V} \leq \int_{\underline{V}}^{\overline{V}} \tilde{V} f(\tilde{V}|h^{t-1}) d\tilde{V} \leq \int_{\underline{V}}^{\overline{V}} \tilde{V} f(\tilde{V}|x>0, h^{t-1}) d\tilde{V} \tag{53}
$$

A sufficient condition for this is that the conditional distribution following a buy FOSD the previous distribution FOSD the distribution following a sell. A sufficient condition for this is MLRP. It therefore suffices to show the following are monotonically increasing in $\tilde{V}$:

$$
\begin{aligned}
\frac{f(\tilde{V}|x>0, h^{t-1})}{f(\tilde{V}|h^{t-1})} &\propto \mathbb{P}[x>0|\tilde{V}, h^{t-1}] \\
\frac{f(\tilde{V}|x<0, h^{t-1})}{f(\tilde{V}|h^{t-1})} &\propto \frac{1}{\mathbb{P}[x<0|\tilde{V}, h^{t-1}]}
\end{aligned}
\tag{54}
$$

From part (1), see that $\mathbb{P}[x > 0|\tilde{V}, h^{t-1}]$ is a step function, jumping up at $A(h^{t-1})$. Thus, $\mathbb{P}[x > 0|\tilde{V}, h^{t-1}]$ is monotonically increasing. Moreover, see that $\mathbb{P}[x < 0|\tilde{V}, h^{t-1}, c]$ is a step function, jumping down at $B(h^{t-1}) - c$. Thus, $\mathbb{P}[x < 0|\tilde{V}, h^{t-1}, c]$ is monotonically decreasing. Moreover, this is true pointwise for all c. Thus, we have pointwise MLRP for all c and the result follows.

3. To prove that $N(h^{t-1}) \leq P^*_{t-1}$, I first compute $N(h^{t-1})$:

$$N(h^{t-1}) = \int_{\underline{V}}^{\overline{V}} \tilde{V} f(\tilde{V}|x = 0, h^{t-1}) d\tilde{V} = \int_{\underline{c}}^{\overline{c}} \int_{\underline{V}}^{\overline{V}} \tilde{V} f(\tilde{V}|x = 0, c, h^{t-1}) d\tilde{V} dJ(c) \quad (55)$$

Where the conditional density is given by:

$$f(\tilde{V}|x = 0, c, h^{t-1}) = \frac{\mathbb{P}[x = 0|\tilde{V}, c, h^{t-1}] f(\tilde{V}|h^{t-1})}{\mathbb{P}[x = 0|c, h^{t-1}]} \quad (56)$$

See that these probabilities are given by:

$$\begin{aligned}
&\mathbb{P}[x = 0|\tilde{V} \geq A(h^{t-1}), c, h^{t-1}] = \pi\delta(h^{t-1}) \\
&+ (1-\pi)(1 - (1 - \delta(h^{t-1}))(1 - K(A(h^{t-1}) - P^*_{t-1})) - (1 - \delta(h^{t-1}))K(B(h^{t-1}) - c - P^*_{t-1})) \\
&\mathbb{P}[x = 0|\tilde{V} \in (B(h^{t-1}) - c, A(h^{t-1})), c, h^{t-1}] = \pi \\
&+ (1-\pi)(1 - (1 - \delta(h^{t-1}))(1 - K(A(h^{t-1}) - P^*_{t-1})) - (1 - \delta(h^{t-1}))K(B(h^{t-1}) - c - P^*_{t-1})) \\
&\mathbb{P}[x = 0|\tilde{V} \leq B(h^{t-1}) - c, c, h^{t-1}] = \pi\delta(h^{t-1}) \\
&+ (1-\pi)(1 - (1 - \delta(h^{t-1}))(1 - K(A(h^{t-1}) - P^*_{t-1})) - (1 - \delta(h^{t-1}))K(B(h^{t-1}) - c - P^*_{t-1}))
\end{aligned} \quad (57)$$

Thus the unconditional probability is given by:

$$\begin{aligned}
&\mathbb{P}[x = 0|c, h^{t-1}] = \pi(1 - \delta(h^{t-1}))(F(A(h^{t-1})|h^{t-1}) - F(B(h^{t-1}) - c|h^{t-1})) \\
&+ \pi\delta(h^{t-1}) + (1-\pi)(1 - (1 - \delta(h^{t-1}))(1 - K(A(h^{t-1}) - P^*_{t-1})) \\
&- (1 - \delta(h^{t-1}))K(B(h^{t-1}) - c - P^*_{t-1}))
\end{aligned} \quad (58)$$

The no-trade price is then given by:

$$N(h^{t-1}) = \frac{\pi(1 - \delta(h^{t-1})) \int_{\underline{c}}^{\overline{c}} \int_{\min\{B(h^{t-1}) - c, \underline{V}\}}^{A(h^{t-1})} \tilde{v} f(\tilde{v}|h^{t-1}) d\tilde{v} dJ(c) + P^*_{t-1}\Xi}{\pi(1 - \delta(h^{t-1}))\mathbb{E}_J[\mathbb{P}[V \in (B(h^{t-1}) - c, A(h^{t-1}))]] + \Xi} \quad (59)$$

where:

$$\Xi = \pi\delta(h^{t-1}) + (1-\pi)\Big[\delta(h^{t-1}) + (1-\delta(h^{t-1}))\Big(1 - (1 - K(A(h^{t-1}) - P^*_{t-1})) - \mathbb{E}_J[K(B(h^{t-1}) - P^*_{t-1} - c)]\Big)\Big] \tag{60}$$

To show that $N(h^{t-1}) \leq P^*_{t-1}$ it therefore suffices to show:

$$\int_{\underline{c}}^{\overline{c}} \int_{\min\{B(h^{t-1})-c,\underline{V}\}}^{A(h^{t-1})} \tilde{v} f(\tilde{v}|h^{t-1}) d\tilde{v} dJ(c) \leq \mathbb{E}_J[\mathbb{P}[V \in (B(h^{t-1}) - c, A(h^{t-1}))]] P^*_{t-1} \tag{61}$$

Or:

$$\mathbb{E}_J[\mathbb{E}[V|V \in (B(h^{t-1}) - c, A(h^{t-1}))]] \leq P^*_{t-1} = \mathbb{E}[V] \tag{62}$$

which is immediately satisfied if $F(\cdot|h^{t-1})$ satisfies sufficient left-tail mass at $(B(h^{t-1}), A(h^{t-1}))$.

Definition 2 (Sufficient Left-Tail Mass)**.** *Given a distribution of short-selling costs J, a distribution F of valuations V satisfies sufficient left-tail mass at (x, y) for $x, y \in [\underline{V}, \overline{V}]$ and $x \leq y$ if:*

$$\mathbb{E}_J\left[\mathbb{E}_F[V|V \in (x - c, y)]\right] \leq \mathbb{E}_F[V] \tag{63}$$

This condition is always satisfied when $V \in \{0, 1\}$. This completes the proof.

4. I prove this result for the richer case that V has CDF F on $[\underline{V}, \overline{V}]$ with continuous PDF f. In the binary case, all integrals over V can be replaced with the conditional probability that $V = 1$ and all results carry. From the proof of Proposition 1, recall that the no-trade price is given by:

$$N(h^{t-1}) = \frac{\pi(1-\delta(h^{t-1})) \int_{\underline{c}}^{\overline{c}} \int_{\min\{B(h^{t-1})-c,\underline{V}\}}^{A(h^{t-1})} \tilde{v} f(\tilde{v}|h^{t-1}) d\tilde{v} dJ(c) + P^*_{t-1}\Xi}{\pi(1-\delta(h^{t-1}))\mathbb{E}_J[\mathbb{P}[V \in (B(h^{t-1}) - c, A(h^{t-1}))]] + \Xi} \tag{64}$$

where:

$$\Xi = \pi\delta(h^{t-1}) + (1-\pi)\Big[\delta(h^{t-1}) + (1-\delta(h^{t-1}))\Big(1 - (1 - K(A(h^{t-1}) - P^*_{t-1})) - \mathbb{E}_J[K(B(h^{t-1}) - P^*_{t-1} - c)]\Big)\Big] \tag{65}$$

See that this can be expressed as a weighted average:

$$N(h^{t-1}) = \frac{\pi(1-\delta(h^{t-1}))\mathbb{E}_J[\mathbb{P}[V \in (B(h^{t-1}) - c, A(h^{t-1}))]]}{\pi(1-\delta(h^{t-1}))\mathbb{E}_J[\mathbb{P}[V \in (B(h^{t-1}) - c, A(h^{t-1}))]] + \Xi}\mathbb{E}_J[\mathbb{E}[V|V \in (B(h^{t-1}) - c, A(h^{t-1}))]] + \frac{\Xi}{\pi(1-\delta(h^{t-1}))\mathbb{E}_J[\mathbb{P}[V \in (B(h^{t-1}) - c, A(h^{t-1}))]] + \Xi} P^*_{t-1} \tag{66}$$

Note also that under the assumption that $F(\cdot|h^{t-1})$ satisfies strictly sufficient left-tail mass at $(B(h^{t-1}), A(h^{t-1}))$ we have:

$$\mathbb{E}_J[\mathbb{E}[V|V \in (B(h^{t-1}) - c, A(h^{t-1}))]] < P^*_{t-1} \tag{67}$$

and observe finally that neither term in the above expression has any dependence on $\delta(h^{t-1})$. To prove the result, it therefore suffices to show that the weight on P^*_{t-1} is increasing in $\delta(h^{t-1})$, that is:

$$\frac{\partial}{\partial \delta(h^{t-1})} \left[\frac{\Xi}{\pi(1 - \delta(h^{t-1}))\mathbb{E}_J[\mathbb{P}[V \in (B(h^{t-1}) - c, A(h^{t-1}))]] + \Xi} \right] \geq 0 \tag{68}$$

To show this, define the following notation:

$$\begin{aligned} \Delta &= K(A(h^{t-1}) - P^*_{t-1}) - \mathbb{E}_J[K(B(h^{t-1}) - P^*_{t-1} - c)] \\ \Lambda &= \mathbb{E}_J[\mathbb{P}[V \in (B(h^{t-1}) - c, A(h^{t-1}))]] \end{aligned} \tag{69}$$

We want to show that:

$$\frac{\partial}{\partial \delta} \left[\frac{\delta + (1 - \pi)\Delta(1 - \delta)}{\pi\Lambda(1 - \delta) + \delta + (1 - \pi)\Delta(1 - \delta)} \right] \geq 0 \tag{70}$$

and we note that $\Delta, \Lambda \in [0, 1]$. The above holds when:

$$\frac{\partial}{\partial \delta} \log\left(\delta + (1 - \pi)\Delta(1 - \delta)\right) \geq \frac{\partial}{\partial \delta} \log\left(\pi\Lambda(1 - \delta)\right) \tag{71}$$

Noting that the function being differentiated on the LHS in monotone increasing in δ as $0 \leq (1 - \pi)\Delta \leq 1$ and the function being differentiated on the RHS is monotone decreasing immediately establishes the result.

□

A.4 Proof of Proposition 2

Proof. We wish to derive a system of difference equations characterizing the evolution of shadow prices. First, note that:

$$P^*_t(x, h^{t-1}) = \mathbb{E}[V|x, h^{t-1}] \tag{72}$$

Applying Bayes' rule and noting that $V \in \{0, 1\}$:

$$\begin{aligned} P_t^*(x, h^{t-1}) &= \mathbb{P}[V = 1 | x, h^{t-1}] \\ &= \frac{\mathbb{P}[x|h^{t-1}, V = 1]\mathbb{P}[V = 1|h^{t-1}]}{\mathbb{P}[x|h^{t-1}, V = 1]\mathbb{P}[V = 1|h^{t-1}] + \mathbb{P}[x|h^{t-1}, V = 0]\mathbb{P}[V = 0|h^{t-1}]} \\ &= \frac{\mathbb{P}[x|h^{t-1}, V = 1]P_{t-1}^*}{\mathbb{P}[x|h^{t-1}, V = 1]P_{t-1}^* + \mathbb{P}[x|h^{t-1}, V = 0](1 - P_{t-1}^*)} \end{aligned} \tag{73}$$

Now see that:

$$1 - P_t^*(x, h^{t-1}) = \frac{\mathbb{P}[x|h^{t-1}, V = 0](1 - P_{t-1}^*)}{\mathbb{P}[x|h^{t-1}, V = 1]P_{t-1}^* + \mathbb{P}[x|h^{t-1}, V = 0](1 - P_{t-1}^*)} \tag{74}$$

Taking the ratio of these two expressions yields the odds-ratio form of shadow prices:

$$\frac{P_t^*(x, h^{t-1})}{1 - P_t^*(x, h^{t-1})} = \frac{P_{t-1}^*}{1 - P_{t-1}^*} \frac{\mathbb{P}[x|V = 1, h^{t-1}]}{\mathbb{P}[x|V = 0, h^{t-1}]} \tag{75}$$

To characterize the difference equation for shadow prices we now need an (i) initial condition $P_{-1}^* = p$ and (ii) the six numbers $\mathbb{P}[x|V = v, h^{t-1}]$ for $x \in \{B, S, N\}$ and $V \in \{0, 1\}$ as a function of h^{t-1}. These numbers are simply given by:

$$\begin{aligned} \mathbb{P}[B|V = 1, h^{t-1}] &= (1 - \delta(h^{t-1}))\left[\pi + (1 - \pi)(1 - K(A(h^{t-1}) - P_{t-1}^*))\right] \\ \mathbb{P}[B|V = 0, h^{t-1}] &= (1 - \delta(h^{t-1}))\left[(1 - \pi)(1 - K(A(h^{t-1}) - P_{t-1}^*))\right] \\ \mathbb{P}[S|V = 1, h^{t-1}] &= (1 - \delta(h^{t-1}))\left[(1 - \pi)\mathbb{E}_c[K(B(h^{t-1}) - P_{t-1}^* - c)]\right] \\ \mathbb{P}[S|V = 0, h^{t-1}] &= (1 - \delta(h^{t-1}))\Big[\pi J(B(h^{t-1})) + (1 - \pi)\mathbb{E}_c[K(B(h^{t-1}) - P_{t-1}^* - c)]\Big] \end{aligned} \tag{76}$$

the no-trade probabilities are given by the residual. Transition probabilities are then given by the law of iterated expectations. More precisely:

$$\begin{aligned} \mathbb{P}[x|h^{t-1}] &= \mathbb{P}[V = 1|h^{t-1}]\mathbb{P}[x|V = 1, h^{t-1}] + \mathbb{P}[V = 0|h^{t-1}]\mathbb{P}[x|V = 0, h^{t-1}] \\ &= P_{t-1}^*\mathbb{P}[x|V = 1, h^{t-1}] + (1 - P_{t-1}^*)\mathbb{P}[x|V = 0, h^{t-1}] \end{aligned} \tag{77}$$

for $x \in \{B, S, N\}$. This completes the proof. □

A.5 Proof of Lemma 1

Proof. I prove the result item by item.

1. To prove the first part of the proposition, I show for extreme valuations that the ask function is concave and the bid function is convex. Applying Proposition 2 we know

that:

$$\begin{aligned}\frac{A(h^{t-1})}{1-A(h^{t-1})} &= \Omega_{t-1}\lambda_A(h^{t-1}) \\ \frac{B(h^{t-1})}{1-B(h^{t-1})} &= \Omega_{t-1}\lambda_B(h^{t-1})\end{aligned} \tag{78}$$

where $\Omega_{t-1} = \frac{P^*_{t-1}}{1-P^*_{t-1}}$ and:

$$\begin{aligned}\lambda_A(h^{t-1}) &= \frac{\mathbb{P}[B|V=1,h^{t-1}]}{\mathbb{P}[B|V=0,h^{t-1}]} \\ \lambda_B(h^{t-1}) &= \frac{\mathbb{P}[S|V=1,h^{t-1}]}{\mathbb{P}[S|V=0,h^{t-1}]}\end{aligned} \tag{79}$$

and:

$$\begin{aligned}\mathbb{P}[B|V=1,h^{t-1}] &= (1-\delta(h^{t-1}))(\pi+(1-\pi)\gamma) \\ \mathbb{P}[B|V=0,h^{t-1}] &= (1-\delta(h^{t-1}))(1-\pi)\gamma \\ \mathbb{P}[S|V=1,h^{t-1}] &= (1-\delta(h^{t-1}))(1-\pi)(1-\gamma) \\ \mathbb{P}[S|V=0,h^{t-1}] &= (1-\delta(h^{t-1}))(\pi+(1-\pi)(1-\gamma))\end{aligned} \tag{80}$$

It follows that the price following $x \in \{B,S\}$ is given by:

$$P_x = \frac{1}{1+\frac{1}{\lambda_x}\frac{1-P^*_{t-1}}{P^*_{t-1}}} \tag{81}$$

where we want to show that P_x is concave in P^*_{t-1} for $\lambda_x > 1$ and convex in P^*_{t-1} for $\lambda_x < 1$. See that this can be rewritten as:

$$P_x\left(1-\frac{1}{\lambda_x}\right)+\frac{P_x}{\lambda_x P^*_{t-1}} = 1 \tag{82}$$

Thus:

$$P_{xP}\left(1-\frac{1}{\lambda_x}\right)+\frac{P_{xP}}{\lambda_x P^*_{t-1}}-\frac{P_x}{\lambda_x P^{*2}_{t-1}} = 0 \tag{83}$$

and:

$$P_{xPP}\left(1+\frac{1}{\lambda_x}\left(\frac{1}{P^*_{t-1}}-1\right)\right)-2\frac{P_{xP}}{\lambda_x P^{*2}_{t-1}}+2\frac{P_x}{\lambda_x P^{*3}_{t-1}} = 0 \tag{84}$$

rearranging this we obtain:

$$P_{xPP}\left(1+\frac{1}{\lambda_x}\left(\frac{1}{P^*_{t-1}}-1\right)\right)=\frac{2}{\lambda_x P^{*2}_{t-1}}\left(P_{xP}-\frac{P_x}{P^*_{t-1}}\right) \tag{85}$$

Note that $\left(1+\frac{1}{\lambda_x}\left(\frac{1}{P^*_{t-1}}-1\right)\right)>0$ and $\frac{2}{\lambda_x P^{*2}_{t-1}}>0$. Thus:

$$\text{sign}(P_{xPP})=\text{sign}\left(P_{xP}-\frac{P_x}{P^*_{t-1}}\right) \tag{86}$$

See also that:

$$\begin{aligned}P_{xP}-\frac{P_x}{P^*_{t-1}}&=P_{xP}-\lambda_x P^*_{t-1}\frac{P_x}{\lambda_x P^{*2}_{t-1}}\\&=P_{xP}-\lambda_x P^*_{t-1}P_{xP}\left(1+\frac{1}{\lambda_x}\left(\frac{1}{P^*_{t-1}}-1\right)\right)\\&=P_{xP}P^*_{t-1}(1-\lambda_x)\end{aligned} \tag{87}$$

Thus, we have the result:

$$\lambda_x>1 \implies P_{xPP}<0 \quad \text{and} \quad \lambda_x<1 \implies P_{xPP}>0 \tag{88}$$

completing the result.

2. If A is concave in P^*_{t-1} and B is convex in P^*_{t-1}, then we have the following for the bid-ask spread:

$$\begin{aligned}BA_t&=A_t-B_t\geq 0\\\frac{\partial^2 BA_t}{\partial {P^*_{t-1}}^2}&=\frac{\partial^2 A_t}{\partial {P^*_{t-1}}^2}-\frac{\partial^2 B_t}{\partial {P^*_{t-1}}^2}\leq 0\end{aligned} \tag{89}$$

Moreover, we know when $P^*_{t-1}=0,1$ that $BA_t=0$. Thus, there must exist a $P'(h^{t-1})$ such that BA_t is decreasing in P^*_{t-1} for all $P^*_{t-1}>P'(h^{t-1})$ and increasing in P^*_{t-1} for all $P^*_{t-1}<P'(h^{t-1})$.

□

A.6 Proof of Proposition 3

Proof. See that the probability of no trade following h^{t-2} and (N, h^{t-2}) is given by:

$$\begin{aligned}\mathbb{P}[N|h^{t-2}] &= P^*_{t-2}\mathbb{P}[N|V=1,h^{t-2}] + (1-P^*_{t-2})\mathbb{P}[N|V=0,h^{t-2}] \\ \mathbb{P}[N|(N,h^{t-2})] &= N(h^{t-2})\mathbb{P}[N|V=1,(N,h^{t-2})] + (1-N(h^{t-2}))\mathbb{P}[N|V=0,(N,h^{t-2})]\end{aligned} \tag{90}$$

Note by Proposition 1 that $N(h^{t-2}) \leq P^*_{t-2}$. Moreover, using the derived transition probabilities from Proposition 2, see that:

$$\begin{aligned}\mathbb{P}[N|V=1,h^{t-2}] = 1-(1-\delta(h^{t-2}))\Big[\pi + (1-\pi)\Big(1-K(A(h^{t-2})-P^*_{t-2})\\ + \mathbb{E}_c[K(B(h^{t-2})-P^*_{t-2}-c)]\Big)\Big]\\ \mathbb{P}[N|V=0,h^{t-2}] = 1-(1-\delta(h^{t-2}))\Big[\pi J(B(h^{t-2})) + (1-\pi)\Big(1-K(A(h^{t-2})-P^*_{t-2})\\ + \mathbb{E}_c[K(B(h^{t-2})-P^*_{t-2}-c)]\Big)\Big]\end{aligned} \tag{91}$$

Hence:

$$\mathbb{P}[N|V=1,h^{t-2}] < \mathbb{P}[N|V=0,h^{t-2}] \quad \forall h^{t-2} \tag{92}$$

Hence, proving that $\mathbb{P}[N|(N,h^{t-2})] > \mathbb{P}[N|h^{t-2}]$, reduces to proving that the conditional probabilities of no trade are both higher following a further observation of no trade. To this end, see that by Lemma 1 if $P^*_{t-1} > P'(N,h^{t-2})$, then the following are true: (i) $A(N,h^{t-2}) - P^*_{t-1} > A(h^{t-2}) - P^*_{t-2}$ (ii) $B(N,h^{t-2}) - P^*_{t-1} < B(h^{t-2}) - P^*_{t-2}$ (iii) $B(N,h^{t-2}) < B(h^{t-2})$. Thus, for fixed $\delta(N,h^{t-2}) = \delta(h^{t-2})$, we have that:

$$\mathbb{P}[N|V=i,(N,h^{t-2})] > \mathbb{P}[N|V=i,h^{t-2}] \tag{93}$$

for $i \in \{0,1\}$. This concludes the proof. □

A.7 Proof of Proposition 4

Proof. Recall that the probability of k sequential no-trade events $S(k)$ starting at any history h^{t-1} is given by applying Bayes rule iteratively as:

$$\mathbb{P}[S(k)|h^{t-1}] = \prod_{i=1}^{k} \mathbb{P}[N|N^{(k-i)},h^{t-1}] \tag{94}$$

The probability of no trade conditional on h^{t-1} and V is given by:

$$\begin{aligned}
\mathbb{P}_K[N|h^{t-1}, V=1] &= \delta(h^{t-1}) + (1-\delta(h^{t-1}))(1-\pi)\Delta_K(h^{t-1}) \\
\mathbb{P}_K[N|h^{t-1}, V=0] &= \delta(h^{t-1}) + (1-\delta(h^{t-1}))\left[(1-\pi)\Delta_K(h^{t-1}) + \pi(1-J(B_K(h^{t-1})))\right]
\end{aligned} \tag{95}$$

See that under K', we have:

$$\Delta_{K'}(h^{t-1}) = K'(A(h^{t-1}) - P^*_{t-1}) - \mathbb{E}_J[K'(B_{K'}(h^{t-1}) - P^*_{t-1} - c)] = 0 \tag{96}$$

Thus the probabilities of $S(k)$ conditional on the state respectively are:

$$\begin{aligned}
\mathbb{P}_K[S(k)|h^{t-1}, V=1] &= \prod_{i=1}^{k}\left[\delta(N^{(k-i)}, h^{t-1}) + (1-\delta(N^{(k-i)}, h^{t-1}))(1-\pi)\Delta_K(N^{(k-i)}, h^{t-1})\right] \\
\mathbb{P}_K[S(k)|h^{t-1}, V=0] &= \prod_{i=1}^{k}\Big[\delta(N^{(k-i)}, h^{t-1}) + (1-\delta(N^{(k-i)}, h^{t-1}))\Big[(1-\pi)\Delta_K(N^{(k-i)}, h^{t-1}) \\
&\quad + \pi(1-J(B_K(N^{(k-i)}, h^{t-1})))\Big]\Big]
\end{aligned} \tag{97}$$

$$\begin{aligned}
\mathbb{P}_{K'}[S(k)|h^{t-1}, V=1] &= \prod_{i=1}^{k}\left[\delta(N^{(k-i)}, h^{t-1})\right] \\
\mathbb{P}_{K'}[S(k)|h^{t-1}, V=0] &= \prod_{i=1}^{k}\left[\delta(N^{(k-i)}, h^{t-1}) + (1-\delta(N^{(k-i)}, h^{t-1}))\pi(1-J(B_{K'}(N^{(k-i)}, h^{t-1})))\right]
\end{aligned} \tag{98}$$

And the unconditional probability of $S(k)$ is given by:

$$\mathbb{P}[S(k)|h^{t-1}] = P^*_{t-1}\mathbb{P}[S(k)|h^{t-1}, V=1] + (1-P^*_{t-1})\mathbb{P}[S(k)|h^{t-1}, V=0] \tag{99}$$

Thus to establish $\mathbb{P}_K[S(k)|h^{t-1}] \geq \mathbb{P}_{K'}[S(k)|h^{t-1}]$ it is sufficient to show:

$$\begin{aligned}
\mathbb{P}_K[S(k)|h^{t-1}, V=1] \geq \mathbb{P}_{K'}[S(k)|h^{t-1}, V=1] \\
\mathbb{P}_K[S(k)|h^{t-1}, V=0] \geq \mathbb{P}_{K'}[S(k)|h^{t-1}, V=0]
\end{aligned} \tag{100}$$

As the $\delta(h^{t-1})$ process is identical under K and K', it is immediate from Equation 97 that $\mathbb{P}_K[S(k)|h^{t-1}, V=1] \geq \mathbb{P}_{K'}[S(k)|h^{t-1}, V=1]$. Finally, it suffices to note that $B_K(h^{t-1}) < B_{K'}(h^{t-1})$ as adverse selection is mechanically worse under K than K' to establish that $\pi(1-$

$J(B_K(N^{(k-i)}, h^{t-1}))) \geq \pi(1 - J(B_{K'}(N^{(k-i)}, h^{t-1})))$ and therefore that $\mathbb{P}_K[S(k)|h^{t-1}, V = 0] \geq \mathbb{P}_{K'}[S(k)|h^{t-1}, V = 0]$, yielding the result. □

A.8 Proof of Proposition 5

Proof. Under the assumed specification for J, we have that $J(B) = 1 - \lambda$ for all $B \in [0, 1]$. Following the first step in Proposition 4, it then suffices to show that:

$$\begin{aligned} \mathbb{P}_J[S(k)|h^{t-1}, V = 1] < \mathbb{P}_{J'}[S(k)|h^{t-1}, V = 1] \\ \mathbb{P}_J[S(k)|h^{t-1}, V = 0] < \mathbb{P}_{J'}[S(k)|h^{t-1}, V = 0] \end{aligned} \tag{101}$$

To show this, it suffices to show that:

$$\Delta_J(N^i, h^{t-1}) < \Delta_{J'}(N^i, h^{t-1}) \quad 0 \leq i \leq k \tag{102}$$

where under the uniform assumption for trader valuations and two point distribution for shorting costs:

$$\Delta(h^{t-1}) = \frac{\lambda(A(h^{t-1}) - P^*_{t-1} - \underline{\eta}) + (1-\lambda)BA(h^{t-1})}{\overline{\eta} - \underline{\eta}} \tag{103}$$

Moreover, neither bid nor ask prices are a function of λ. Thus, $\Delta(h^{t-1})$ is increasing in λ. Further, conditional on $\Delta(h^{t-1})$, the no-trade price is a decreasing function of λ:

$$N(h^{t-1}) = \frac{P^*_{t-1}\left(\delta(h^{t-1}) + (1-\delta(h^{t-1}))(1-\pi)\Delta(h^{t-1})\right)}{\delta(h^{t-1}) + (1-\delta(h^{t-1}))\left[(1-\pi)\Delta(h^{t-1}) + \pi\lambda(1-P^*_{t-1})\right]} \tag{104}$$

We therefore have that for all $0 \leq i \leq k$, $N_{\lambda'}(N^i, h^{t-1}) \geq N_\lambda(N^i, h^{t-1})$ whenever π is sufficiently large. Moreover, if $P^*_{\lambda,t-1+i}, P^*_{\lambda',t-1+i} > \max\{P'_\lambda(h^{t-1}), P'_{\lambda'}(h^{t-1})\}$ we have that $BA_{\lambda'}(N^i, h^{t-1}) \leq BA_\lambda(N^i, h^{t-1})$ and $A_{\lambda'}(N^i, h^{t-1}) - P^*_{\lambda'}(N^i, h^{t-1}) \leq A_\lambda(N^i, h^{t-1}) - P^*_\lambda(N^i, h^{t-1})$. This proves the required inequality for all $k \leq \bar{k} = \max\{i | P^*_{\lambda,t-1+i}, P^*_{\lambda',t-1+i} > \max\{P'_\lambda(h^{t-1}), P'_{\lambda'}(h^{t-1})\}\}$ and completes the proof. □

A.9 Proof of Proposition 6

Proof. Recall:

$$\mathbb{P}[S(k)|h^{t-1}] = P^*_{t-1}\mathbb{P}[S(k)|h^{t-1}, V = 1] + (1 - P^*_{t-1})\mathbb{P}[S(k)|h^{t-1}, V = 0] \tag{105}$$

and Equation 97. Thus, it suffices to prove that:

$$\begin{aligned}\mathbb{P}_K[S(k)_\xi|h^{t-1}, V = 1] < \mathbb{P}_K[S(k)_0|h^{t-1}, V = 1] \\ \mathbb{P}_K[S(k)_\xi|h^{t-1}, V = 0] < \mathbb{P}_K[S(k)_0|h^{t-1}, V = 0]\end{aligned} \tag{106}$$

Note that $\Delta(N^{k-i}, h^{t-1}) = 0$ and $J(B(N^{k-i}, h^{t-1}))$ is not a function of B given our assumptions on short-selling costs and noise trader valuations. Moreover, we have that:

$$\delta(h^{t-1})_\xi < \delta(h^{t-1})_0 \tag{107}$$

The result follows immediately. □

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Online Appendix to Dynamic Unravelling

B Policy, Price Discovery, and Welfare

In this Appendix, I show how short-selling constraints and position limits affect price discovery. I moreover derive the model's implied welfare metric and show how preventing low-volume crashes and improving price discovery emerge endogenously as desiderata on the part of the policymaker, justifying my earlier taking of these as reduced-form goals.

B.1 Policy and Price Discovery

Having shown the impact of short-selling constraints and position limits on the likelihood of market crashes, it is further instructive and important for policy to consider the impact of these policies on price discovery.

For the case of short-selling constraints, my analysis simply echoes the result of Diamond and Verrecchia (1987) in my setting: short-selling constraints harm price discovery. In addition to this, I also show that position limits harm price discovery in my model. The intuition for both results is simple: short-selling constraints and position limits both impede the flow of information from informed traders into prices.

I define price discovery analogously to Diamond and Verrecchia (1987). Consider two price thresholds $\underline{P} < P^*_{-1} = p < \overline{P}$. Define $\overline{T}$ as the earliest time at which $P^*_t \geq \overline{P}$ or $P^*_t \leq \underline{P}$. More formally:

$$\overline{T} = \begin{cases} \min\{\ t \mid P^*_t \geq \overline{P} \text{ or } P^*_t \leq \underline{P}\}, & \text{if this exists,} \\ T, & \text{otherwise.} \end{cases} \tag{108}$$

We will call $\overline{T}$ the price discovery time. We are now ready to define formally our meaning of faster (slower) price discovery under various parameterizations of the model:

Definition 3 (Price Discovery)**.** *Given $(\underline{P}, \overline{P})$, under two parameterizations of the model Λ, Λ', price discovery is slower under Λ' than Λ in state of the world $V = i$ if:*

$$T_i^{\Lambda'} \geq T_i^{\Lambda} \tag{109}$$

where $T_i^{\Lambda} = \mathbb{E}[\overline{T}|V = i]$.

We are now in a position to ask how short-selling costs and position limits affect price discovery in the model. Toward this, recall the likelihood ratio property established earlier.

$$\Omega_{\overline{T}} = \Omega_{\overline{T}-1}\frac{\mathbb{P}[X|V=1,h^{t-1}]}{\mathbb{P}[X|V=0,h^{t-1}]} \tag{110}$$

where $\Omega_t \equiv \frac{P_t^*}{1-P_t^*}$ is the odds ratio. Rearranging this expression yields:

$$\log\Omega_{\overline{T}} = \log\Omega_{\overline{T}-1} + \log\frac{\mathbb{P}[X|V=1,h^{t-1}]}{\mathbb{P}[X|V=0,h^{t-1}]} \tag{111}$$

This is instructive as it tells us that the log-odds ratio follows a random walk, where innovations are given by the relative log ratio of the probability that the signal received at date t was generated in the state of the world where V is high versus the state of the world where V is low. We can therefore iterate Equation (111) to obtain the following expression for the log-odds ratio:

$$\log\Omega_{\overline{T}} = \sum_{t=0}^{\overline{T}}\log\frac{\mathbb{P}[X|V=1,h^{t-1}]}{\mathbb{P}[X|V=0,h^{t-1}]} \tag{112}$$

Given that $\overline{T}$ is a stopping time (see Equation (108)), one can apply Wald's equation to yield:

$$\mathbb{E}[\log\Omega_{\overline{T}}] = \mathbb{E}\Big[\sum_{t=0}^{\overline{T}}\mathbb{E}\Big[\log\frac{\mathbb{P}[X|V=1,h^{t-1}]}{\mathbb{P}[X|V=0,h^{t-1}]}\Big]\Big] \tag{113}$$

To operationalize the use of Wald's equation, it is further necessary to remove the history dependence from the above expression. To this end, I make the following assumption which is necessary for this result only:

Assumption 1. *The following hold:*

1. $G(\cdot|h^{t-1}) \equiv G$ – *book capacity is history independent*
2. $c = \overline{c} > \max\{1, |\underline{\eta}|\}$ *with probability* λ *and* $c = 0$ *with probability* $1-\lambda$ – *with probability* λ *short-selling is prohibitively costly and free otherwise*
3. $\eta = \underline{\eta} < -1$ *with probability* $1-\gamma$ *and* $\eta = \overline{\eta} > 1$ *with probability* γ – *noise traders mechanically always wish to either buy or sell, independently of the current bid-ask spread.*

Under Assumption 1, this expression becomes history independent and we have the following tractable expression for the average price discovery time:

$$\mathbb{E}[\overline{T}] = \frac{\mathbb{E}[\log\Omega_{\overline{T}}]}{\mathbb{E}\Big[\log\frac{\mathbb{P}[X|V=1]}{\mathbb{P}[X|V=0]}\Big]} \tag{114}$$

To perform comparative statics on price discovery, it therefore suffices to perform comparative statics on Equation 114. Performing this exercise for the distribution of position limits and short-selling constraints yields Proposition 7:

Proposition 7. *If Assumption 1 holds, then:*

1. *If two short-selling cost distributions are such that $J' \succ_{FOSD} J$, then under J' price discovery is slower than under J in both states of the world:*

$$T_1' \geq T_1 \quad \textit{and} \quad T_0' \geq T_0 \tag{115}$$

2. *If two position limit distributions are such that $G' \succ_{FOSD} G$. Under G' price discovery is faster than under G in both states of the world:*

$$T_1' \leq T_1 \quad \textit{and} \quad T_0' \leq T_0 \tag{116}$$

Proof. Recall the odds-ratio property of prices established earlier:

$$\Omega_{\overline{T}} = \Omega_{\overline{T}-1} \frac{\mathbb{P}[X|V=1, h^{t-1}]}{\mathbb{P}[X|V=0, h^{t-1}]} \tag{117}$$

where $\Omega_t \equiv \frac{P_t^*}{1-P_t^*}$ is the odds ratio. Rearranging this expression yields:

$$\log \Omega_{\overline{T}} = \log \Omega_{\overline{T}-1} + \log \frac{\mathbb{P}[X|V=1, h^{t-1}]}{\mathbb{P}[X|V=0, h^{t-1}]} \tag{118}$$

Iterating:

$$\log \Omega_{\overline{T}} = \sum_{t=0}^{\overline{T}} \log \frac{\mathbb{P}[X|V=1, h^{t-1}]}{\mathbb{P}[X|V=0, h^{t-1}]} \tag{119}$$

Given that $\overline{T}$ is a stopping time (see Equation (108)), one can apply Wald's equation to yield:

$$\mathbb{E}[\log \Omega_{\overline{T}}] = \mathbb{E}\Big[\sum_{t=0}^{\overline{T}} \mathbb{E}\Big[\log \frac{\mathbb{P}[X|V=1, h^{t-1}]}{\mathbb{P}[X|V=0, h^{t-1}]}\Big]\Big] \tag{120}$$

If one further imposes Assumption 1, the probabilities derived in Proposition 4 can be simplified:

$$
\begin{aligned}
\mathbb{P}[B|V=1,h^{t-1}] &= (1-\delta(h^{t-1}))\left[\pi + (1-\pi)(1-K(A(h^{t-1})-P^*_{t-1}))\right] \\
&= (1-\delta)(\pi+(1-\pi)\gamma) \\
\mathbb{P}[B|V=0,h^{t-1}] &= (1-\delta(h^{t-1}))\left[(1-\pi)(1-K(A(h^{t-1})-P^*_{t-1}))\right] \\
&= (1-\delta)(1-\pi)\gamma \\
\mathbb{P}[S|V=1,h^{t-1}] &= (1-\delta(h^{t-1}))\left[(1-\pi)\mathbb{E}_c[K(B(h^{t-1})-P^*_{t-1}-c)]\right] \\
&= (1-\delta)(1-\pi)(1-\gamma)(1-\lambda) \\
\mathbb{P}[S|V=0,h^{t-1}] &= (1-\delta(h^{t-1}))\Big[\pi J(B(h^{t-1})) + (1-\pi)\mathbb{E}_c[K(B(h^{t-1})-P^*_{t-1}-c)]\Big] \\
&= (1-\delta)(\pi(1-\lambda)+(1-\pi)(1-\gamma)(1-\lambda))
\end{aligned}
\tag{121}
$$

Which are now all history independent. The stopping time equation can therefore be simplified:

$$\mathbb{E}[\overline{T}] = \frac{\mathbb{E}[\log \Omega_{\overline{T}}]}{\mathbb{E}\Big[\log \frac{\mathbb{P}[X|V=1]}{\mathbb{P}[X|V=0]}\Big]} \tag{122}$$

To establish comparative statics in T_i, see also that:

$$T_i = \frac{\mathbb{E}[\log \Omega_{\overline{T}}|V=i]}{\mathbb{E}\Big[\log \frac{\mathbb{P}[X|V=1]}{\mathbb{P}[X|V=0]}|V=i\Big]} \tag{123}$$

As noted by Diamond and Verrecchia (1987), the numerator can be computed given the observation that we are performing a Wald sequential likelihood test of $H_0 : V = 0$ versus $H_1 : V = 1$ with decision thresholds given by:

$$
\begin{aligned}
\Omega_t &\geq \log \frac{\overline{P}}{1-\overline{P}} \equiv \log B \implies H_1 \text{ True} \\
\Omega_t &\leq \log \frac{\overline{P}}{1-\overline{P}} \equiv \log A \implies H_0 \text{ True}
\end{aligned}
\tag{124}
$$

Now we wish to compute the conditional expectation of the log-likelihood ratio conditional on stopping, which can be shown to be well-approximated by:[21]

$$
\begin{aligned}
\mathbb{E}[\log \Omega_{\overline{T}}|V=0] &\approx \mathbb{P}[\text{Type I Error}]\log B + (1-\mathbb{P}[\text{Type I Error}])\log A \\
\mathbb{E}[\log \Omega_{\overline{T}}|V=1] &\approx (1-\mathbb{P}[\text{Type II Error}])\log B + \mathbb{P}[\text{Type II Error}]\log A
\end{aligned}
\tag{125}
$$

[21]The approximation stems from crossing the barrier rather than hitting the barrier in discrete time.

Letting $\alpha \equiv \mathbb{P}[\text{Type I Error}]$ and $\beta \equiv \mathbb{P}[\text{Type II Error}]$, we can further compute:

$$\begin{aligned} A &= \frac{\beta}{1-\alpha} \\ B &= \frac{1-\beta}{\alpha} \end{aligned} \tag{126}$$

Solving this system and plugging into the above yields:

$$\begin{aligned} \mathbb{E}[\log \Omega_{\overline{T}} | V = 0] &\approx \frac{1-A}{B-A} \log B + \frac{B-1}{B-A} \log A \\ \mathbb{E}[\log \Omega_{\overline{T}} | V = 1] &\approx \frac{B(1-A)}{B-A} \log B + \frac{A(B-1)}{B-A} \log A \end{aligned} \tag{127}$$

Hence, in both cases, the denominator of T_i depends only on $\underline{P}$ and $\overline{P}$. Hence, to consider comparative statics on T_i it suffices to consider only the denominator:

$$D_i \equiv \mathbb{E}\Big[\log \frac{\mathbb{P}[X|V=1]}{\mathbb{P}[X|V=0]} | V = i\Big] \tag{128}$$

In this context, proving the proposition reduces to showing (i) for $J' \succ_{FOSD} J$ that $D_i^{J'} < D_i^J$ and (ii) for $G' \succ_{FOSD} G$ that $D_i^{G'} < D_i^G$. To this end, see that D_i can be written as:

$$D_i = \mathbb{P}[B|V=i] \log \frac{\mathbb{P}[B|V=1]}{\mathbb{P}[B|V=0]} + \mathbb{P}[S|V=i] \log \frac{\mathbb{P}[S|V=1]}{\mathbb{P}[S|V=0]} + \mathbb{P}[N|V=i] \log \frac{\mathbb{P}[N|V=1]}{\mathbb{P}[N|V=0]} \tag{129}$$

This equation is simply a function of the six numbers pinned down in Proposition 4. See that $\delta_{G'} \leq \delta_G$ and $\lambda_{J'} \geq \lambda_J$. The comparative statics follow from plugging in, differentiating and checking that $\frac{\partial D_i}{\partial \delta} \leq 0$ and $\frac{\partial D_i}{\partial \lambda} \leq 0$.

Explicitly, see that D_1 and D_0 are given by:

$$\begin{aligned} D_1 = &(1-\delta)[\pi + (1-\pi)\gamma] \log \frac{\pi + (1-\pi)\gamma}{(1-\pi)\gamma} \\ &+ (1-\delta)[(1-\pi)(1-\gamma)(1-\lambda)] \log \frac{(1-\pi)(1-\gamma)}{\pi + (1-\pi)(1-\gamma)} \\ &+ [1 - (1-\delta)\,[\pi + (1-\pi)[\gamma + (1-\gamma)(1-\lambda)]] \\ &\times \log \frac{1 - (1-\delta)\,[\pi + (1-\pi)[\gamma + (1-\gamma)(1-\lambda)]}{1 - (1-\delta)\,[\pi(1-\lambda) + (1-\pi)[\gamma + (1-\gamma)(1-\lambda)]} \end{aligned} \tag{130}$$

$$
\begin{aligned}
D_0 = {} & (1-\delta)[(1-\pi)\gamma] \log \frac{\pi + (1-\pi)\gamma}{(1-\pi)\gamma} \\
& + (1-\delta)[\pi(1-\lambda) + (1-\pi)(1-\gamma)(1-\lambda)] \log \frac{(1-\pi)(1-\gamma)}{\pi + (1-\pi)(1-\gamma)} \\
& + [1 - (1-\delta)\,[\pi(1-\lambda) + (1-\pi)[\gamma + (1-\gamma)(1-\lambda)]] \\
& \times \log \frac{1 - (1-\delta)\,[\pi + (1-\pi)[\gamma + (1-\gamma)(1-\lambda)]}{1 - (1-\delta)\,[\pi(1-\lambda) + (1-\pi)[\gamma + (1-\gamma)(1-\lambda)]}
\end{aligned}
\tag{131}
$$

Symbolic computation via Mathematica confirms that $\frac{\partial D_i}{\partial \delta} \leq 0$ and $\frac{\partial D_i}{\partial \lambda} \leq 0$, completing the proof.[22] □

I first provide intuition for the fact that tighter short-selling constraints worsen price discovery. This result echoes the one from Diamond and Verrecchia (1987). In particular, when short-selling constraints are tighter there is naturally slower price discovery when the fundamental value of the security is low as informed traders are prevented from selling the security. When the fundamental value of the security is high, tighter short-selling constraints both reduce the fraction of noise traders who sell and worsen inference conditional on observations of no-trade. However, it can be shown that the latter effect dominates so that price discovery is slower. For the positive impact of the position limit distribution on price discovery, see that when limits are looser, no-trade events become less common which aids discovery. This is because the chance an agent makes an order in excess of the market maker's capacity is diminished. At the same time, no-trade events constitute a less negative signal. This second effect aids discovery when the value is high and harms it when the value is low. This notwithstanding, it can be shown that the first effect dominates the latter as the latter is infra-marginal. Thus, increases in position limits and corresponding liquidity injections speed price discovery in the model.

B.2 Policy and Welfare

So far my analysis of policy has taken policymaker objectives to reduce the likelihood of crashes and improve price discovery as given. However, the formalism of the model admits a natural welfare metric whereby the planner's objectives can be endogenized. This allows an analysis of optimal policy, both static and dynamic, from the perspective of the welfare of market participants. Price discovery and preventing crashes emerge as natural desiderata when one uses the implied welfare metric, providing a microfoundation for the earlier, qualitative analysis of policy.

[22] I omit the specific computation of the derivative as it is cumbersome.

B.2.1 The Welfare Metric

As all trades on the basis of information are transfers, welfare in the model stems from the idiosyncratic valuations of noise traders and therefore takes seriously that their wish to trade is optimal. Toward deriving the endogenous welfare metric in the model, I first assume that the policymaker is utilitarian and places equal welfare weights on the expected utility of all market participants. Implicit in this is the assumption that prices or activity in this market have no affect on other (unmodelled) markets or the real economy at large. Formally, the per-period payoff of the designer $W_t : \mathcal{H}^{t-1} \to \mathbb{R}$ is given by:

$$W_t(h^{t-1}) = (1-\pi)\mathbb{E}[U^U|h^{t-1}] + \pi\mathbb{E}[U^I|h^{t-1}] + \mathbb{E}[U^M|h^{t-1}] \tag{132}$$

where U^U, U^I and U^M are the utilities of uninformed traders, informed traders and market makers respectively.

Using the previously established structure of equilibrium and the optimal demand policies of traders, this welfare metric can be expressed as a function of model primitives and the history. This is stated formally in Proposition 8.

Proposition 8. *The welfare function of the policymaker is given by:*

$$\begin{aligned} W_t(h^{t-1}) = (1-\pi)(1-\delta(h^{t-1}))|x^*(h^{t-1})| \Bigg[\int_{\underline{c}}^{\overline{c}} \int_{\underline{\eta}}^{B(h^{t-1})-c-P^*_{t-1}} -\eta dK(\eta) dJ(c) \\ + \int_{\underline{c}}^{\overline{c}} \int_{A(h^{t-1})-P^*_{t-1}}^{\overline{\eta}} \eta dK(\eta) dJ(c) \Bigg] \end{aligned} \tag{133}$$

Proof. We have that the payoff function of the policymaker is given by:

$$W_t(h^{t-1}) = (1-\pi)\mathbb{E}[U^U|h^{t-1}] + \pi\mathbb{E}[U^I|h^{t-1}] + \mathbb{E}[U^M|h^{t-1}] \tag{134}$$

Consider state by state any trades that occur. There are two possibilities. First, an informed trader can trade with a market maker. In this case an informed trader transacts x units of the security at price p when the true value is V. The informed trader gets utility $x(V - p - c\mathbb{I}[x < 0]c)$ and the market maker gets utility $x(p - V)$, which nets to zero, modulo short-selling cost. Moreover, the short-selling cost (if incurred) is again just a transfer. Thus, conditional on an informed trade, welfare is zero. Second, an uninformed trader can trade with a market maker. The uninformed trader gets expected utility $x(P^*_{t-1} + \eta - p - c\mathbb{I}[x < 0])$ and the market maker gets expected utility $x(p - P^*_{t-1})$. Again the short-selling cost is simply a transfer. Thus welfare is given by ηx conditional on trade.

Now note that the uninformed agent will wish to sell whenever $\eta \in [\underline{\eta}, B(h^{t-1}) - c - P^*_{t-1}]$

and buy whenever $\eta \in [A(h^{t-1}) - P^*_{t-1}, \overline{\eta}]$. Moreover, conditional on the sign of trade, their demand solves the FOC:

$$(1 - G(|x_t|^*)) \times \text{sign}(x^*_t) - g(|x^*_t|) = 0 \tag{135}$$

Thus, as is shown in the proof of Theorem 1, the magnitude of x_t is not a function of η. The probability of the agent failing to trade is then given by $\delta(h^{t-1}) = G(|x_t|^*)$, which again does not depend on η. Thus, the agent's expected utility is separable in x and η up to the sign of x. Noting that the chance of an uninformed trader arriving to the market is $(1 - \pi)$, it follows that conditional on a given short-selling cost welfare is given by:

$$\begin{aligned} W(h^{t-1}, c) = (1 - \pi)(1 - \delta(h^{t-1}))|x^*(h^{t-1})| \Big[&\int_{\underline{\eta}}^{B(h^{t-1})-c-P^*_{t-1}} -\eta dK(\eta) \\ + \int_{A(h^{t-1})-P^*_{t-1}}^{\overline{\eta}} \eta dK(\eta) \Big] \end{aligned} \tag{136}$$

Averaging over the distribution of short-selling costs then yields welfare:

$$\begin{aligned} W_t(h^{t-1}) = (1 - \pi)(1 - \delta(h^{t-1}))|x^*(h^{t-1})| \Big[&\int_{\underline{c}}^{\overline{c}} \int_{\underline{\eta}}^{B(h^{t-1})-c-P^*_{t-1}} -\eta dK(\eta) dJ(c) \\ + \int_{\underline{c}}^{\overline{c}} \int_{A(h^{t-1})-P^*_{t-1}}^{\overline{\eta}} \eta dK(\eta) dJ(c) \Big] \end{aligned} \tag{137}$$

□

The intuition for this form is that all short-selling costs and trades are simply transfers between agents except for trades undertaken by noise traders on the basis of idiosyncratic valuation. As a result, the welfare metric is simply the expected quantity transacted by noise traders multiplied by the expected idiosyncratic valuation conditional on trade. In the following analysis, I use this welfare metric to perform analytical static welfare analysis and numerical dynamic welfare analysis.

B.2.2 Static Welfare Analysis

With the welfare metric in hand, we can now analyze which features of the environment will give rise to high or low welfare. Toward this, I first provide a simple closed form example for the welfare metric in the case of uniform noise trader demand to illustrate the key trade-offs involved in the policymaker's welfare function.

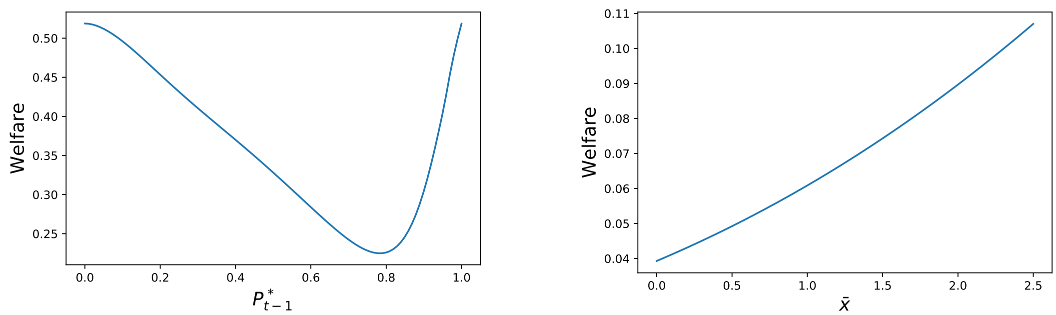


Figure 6: Left: Welfare as a Function of P^*_{t-1}, Right: Welfare as a function of $\bar{x}$. Parameters: $p = 0.7, \pi = 0.5, \eta \sim U[-1, 1], c \sim U[0, 0.8], \varepsilon \sim N(0, 9)$.

Example 1 (Welfare with Uniform Valuations)**.** Suppose that $\eta \sim U[\underline{\eta}, \overline{\eta}]$ with $\underline{\eta} \leq -1 - \overline{c}$ and $\overline{\eta} \geq 1$. Integrating with respect to this density yields:

$$W_t(h^{t-1}) = \underbrace{(1-\pi)(1-\delta(h^{t-1}))x^*(h^{t-1})}_{\text{Expected Quantity Transacted by Noise Traders}} \Bigg[\underbrace{\frac{\overline{\eta}^2 + \underline{\eta}^2}{2(\overline{\eta} - \underline{\eta})}}_{\text{Total Value If All Traded}} \\ - \underbrace{\frac{(A(h^{t-1}) - P^*_{t-1})^2 + \mathbb{E}[(B(h^{t-1}) - c - P^*_{t-1})^2]}{2(\overline{\eta} - \underline{\eta})}}_{\text{Lost Trades Because of Bid-Ask Spread}} \Bigg] \tag{138}$$

The term on the second line clarifies that welfare is driven primarily by trade probabilities and the width of bid-ask spreads, with the former being beneficial for welfare and the latter harmful as more noise traders are unwilling to pay to cross the spread, creating unrealized gains from trade and harming welfare. To show this function numerically, I moreover assume that market maker book capacity is Gaussian and $\bar{x}_t = \bar{x} + \varepsilon_t$ with $\varepsilon_t \sim N(0, \sigma^2_\varepsilon)$. Figure 6 Shows welfare as a function of the previous shadow price and the average market maker book capacity. As the formula shows, welfare is lowest close to the prior as that is where bid-ask spreads are maximized and the unrealized gains from trade are highest. △

These core trade-offs naturally extend to the welfare metric under alternative specifications of noise trader demand. This is stated formally in the following proposition:

Proposition 9. *If (i) $|A(h^{t-1}) - P^*_{t-1}|$ rises (ii) $|P^*_{t-1} - B(h^{t-1})|$ rises or (iii) G_t changes to G'_t such that $G_t \succ_{FOSD} G'_t$, then $W_t(h^{t-1})$ falls*

Proof. The claims all follow by direct inspection of the welfare function in the appropriate objects. I show the claims case-by-case.

1. If $|A(h^{t-1}) - P^*_{t-1}|$ rises, then $W_t(h^{t-1})$ falls:

$$\frac{\partial W_t(h^{t-1})}{\partial |A(h^{t-1}) - P^*_{t-1}|} = -(1-\pi)(1-\delta(h^{t-1}))|x^*(h^{t-1})|k(A(h^{t-1}) - P^*_{t-1}) < 0 \tag{139}$$

2. if $|P^*_{t-1} - B(h^{t-1})|$ rises, then $W_t(h^{t-1})$ falls:

$$\frac{\partial W_t(h^{t-1})}{\partial |P^*_{t-1} - B(h^{t-1})|} = -(1-\pi)(1-\delta(h^{t-1}))|x^*(h^{t-1})| \int_{\underline{c}}^{\bar{c}} k(B(h^{t-1}) - c - P^*_{t-1}) dJ(c) < 0 \tag{140}$$

3. If $G_t \succ_{FOSD} G'_t$, then $W_t(h^{t-1})$ falls. Consider the value function of the noise trader:

$$O(h^{t-1}) \propto (1-\delta(h^{t-1}))|x^*(h^{t-1})| \tag{141}$$

   This must strictly fall when $G_t \succ_{FOSD} G'_t$. Moreover, the bid and ask are unchanged. Thus, $W_t(h^{t-1})$ must fall.

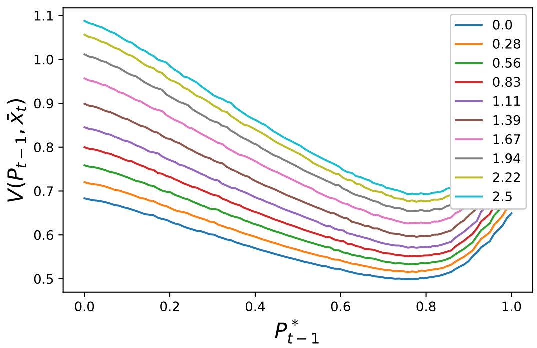
□

This result clarifies that large spreads and small position limits naturally reduce welfare as they limit the volume of welfare-improving trades that can occur. As a result, it provides a natural microfoundation for the earlier qualitative analysis that took it as given that the policymaker wants to reduce the probability of crashes and improve price discovery, as both policies actively reduce spreads and thereby improve welfare.

B.2.3 Dynamic Welfare Analysis

Having understood the static motives for policy, the analysis can be extended to understand how policy affects welfare dynamically. This is interesting as policy in any period t affects the transition probabilities and therefore both future rationing probabilities and the size of the bid-ask spread. I therefore describe a method for computing welfare in a dynamic setting and illustrate the dynamic welfare criterion in a quantitative example.

Mapping to the dynamic context tractably does, however, involve one departure from the model as studied so far. Because the state has so far been persistent, welfare is a function of calendar time – a wholly undesirable feature for a dynamic analysis. I therefore make the problem stationary by assuming that the underlying state V is drawn from $\{0, 1\}$ at date 0 with probability of p that $V = 1$ and that V is then reset at Poisson rate ω and that each reset of V is a commonly known event. Concretely, at the end of each period t, with probability ω, V is drawn from $\{0, 1\}$ with probability p that $V = 1$ and the common prior



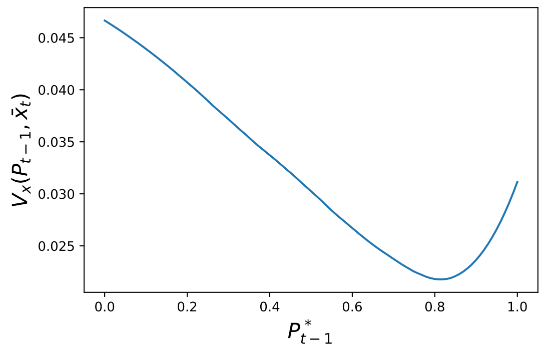


Figure 7: Left: Contour Plot of Value Function (Contours for Differing Values of $\bar{x}_t$), Right: Numerical Derivative of the Value Function as a Function of $\bar{x}_t$ (Averaged over $\bar{x}_t$). Parameters: Parameters: $p = 0.7, \pi = 0.5, \eta \sim U[-1, 1], c \sim U[0, 0.8], \varepsilon \sim N(0, 9), \omega = 0.1, \rho = 0.9$.

resets so that $P^*_{t-1} = p$. The policymaker is assumed to have dicount factor ρ and dynamic preferences given by:

$$V(P^*_{t-1}, \bar{x}_t) = \mathbb{E}\left[\sum_{i=0}^{\infty} \rho^t W(P^*_{t+i-1}, \bar{x}_{t+i}) | P^*_{t-1}, \bar{x}_t\right] \tag{142}$$

This can of course be written recursively as:

$$V(P^*_{t-1}, \bar{x}_t) = W(P^*_{t-1}, \bar{x}_t) + (1-\omega)\rho\mathbb{E}[V(P^*_t, \bar{x}_{t+1}) | P^*_{t-1}, \bar{x}_t] + \omega\rho\mathbb{E}[V(p, \bar{x}_{t+1}) | P^*_{t-1}, \bar{x}_t] \tag{143}$$

Naturally, closed-form computation of this objective is infeasible even in the simplest of circumstances. However, as I have solved for the entire Markov chain describing the model in Proposition 2, computation of V via value function iteration is numerically simple. In particular, one need only specify: i) the short-selling cost distribution J ii) the idiosyncratic valuation distribution K and iii) the Markov process for position limits. In the following example I show one such parameterization and study the implications for the effectiveness of liquidity interventions.

Example 2 (Dynamic Policy Analysis)**.** Suppose that trader valuations are uniform $\eta \sim U[\underline{\eta}, \overline{\eta}]$, short-selling costs are uniform $c \sim U[0, \overline{c}]$ and that book capacity follows a random walk:

$$\bar{x}_t = \bar{x}_{t-1} - |x_{t-1}| + \varepsilon_t \tag{144}$$

where $\varepsilon_t \sim N(0, \sigma^2_\varepsilon)$. The left panel Figure 7 plots welfare as a function of the current price and the size of market maker book capacity. Intuitively, this dynamic analysis echoes the static analysis: welfare is highest when we are close to price discovery and lowest around the prior where adverse selection is worst, spreads are widest and gains from trade are forgone.

Welfare is also naturally increasing in the level of book capacity $\bar{x}_t$ as this increases the potential for welfare-improving trade. Moreover, the right panel of Figure 7 shows that the marginal value of a position limit intervention is greatest closest to price discovery and lowest around the prior. This is intuitive as there are more noise traders in the market who can utilize any liquidity injection when bid-ask spreads are smallest, which occurs around full price discovery. This provides a rationale for micro-prudential policy targeting liquidity interventions in the early stages of a low volume crash to arrest the dynamic unravelling mechanism and ensure tight bid-ask spreads. $\triangle$

The ability to perform such dynamic welfare analysis may be of interest other researchers in the context of these models. One can additionally use the Markov chain structure of the model to compute the stationary distribution of prices and compute counterfactual welfare analyses, of which the position limit intervention given in Example 2 is just one possibility.