Econometrica Supplementary Material

SUPPLEMENT TO "CONTRACTIBILITY DESIGN"

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B. OMITTED PROOFS

B.1. *Proof of Lemma 2*

In Lemma 12 in Appendix C.1, we show that a regular contractibility correspondence C can be equivalently represented via (i) two sets of self-enforcing recommendations $\underline{D}, \overline{D}$ (see Lemma 1 in the main text); or via (ii) $C(y) = [\underline{\delta}(y), \overline{\delta}(y)]$ for all $y \in X$, where $\underline{\delta} : X \to X$ is an upper semi-continuous increasing function and $\overline{\delta} : X \to X$ is a lower semi-continuous increasing function such that: (i) $y \in [\underline{\delta}(y), \overline{\delta}(y)]$, (ii) $\underline{\delta}(x) = \underline{\delta}(y)$ for all $x \in [\underline{\delta}(y), y)$, (iii) $\overline{\delta}(x) = \overline{\delta}(y)$ for all $x \in (y, \overline{\delta}(y)]$, and (iv) $\underline{\delta}(0) = 0$.

There are only two $\underline{\delta}$ functions satisfying the properties of Lemma 1 and such that $\underline{\delta}(X) \subseteq \{0, \overline{x}\}$: (i) $\underline{\delta}(x) = 0$ and (ii) $\underline{\delta}(x) = \overline{x}\mathbb{I}[x = \overline{x}]$. Assume by contradiction that $\underline{D} \nsubseteq \{0, \overline{x}\}$, and therefore $\underline{\delta}$ does not correspond to one of these two functions. Thus, there exists an $x_0 \in (0, \overline{x}]$ such that $\underline{\delta}(x_0) \in (0, \overline{x})$. Consider replacing this $\underline{\delta}$ with $\underline{\delta}' = 0$, defining a corresponding new evidence structure $\mathcal{E}'$. In Lemma 9 in Appendix A.1, we show that implementable actions and conditionally optimal tariffs depend only on the upper self-enforcing recommendations, summarized by the set $\overline{D}$ and function $\overline{\delta}$. Therefore, it suffices to consider the effect on front-end and back-end costs. For front-end costs, we observe (see the proof of Lemma 6 in Appendix A.5) that

$$\Gamma_f(\mathcal{E}) = \int_X \left[\int_0^{\underline{\delta}_{\mathcal{E}}(x)} g_f(x,z)\,\mathrm{d}z + \int_{\overline{\delta}_{\mathcal{E}}(x)}^{\bar{x}} g_f(x,z)\,\mathrm{d}z \right] \mathrm{d}x\,. \tag{67}$$

Considered as a functional of $\underline{\delta}, \overline{\delta}$, Γ_f is monotone. Moreover, $\underline{\delta}_{\mathcal{E}} \geq \underline{\delta}_{\mathcal{E}'}$. Thus, $\Gamma_f(\mathcal{E}) \geq \Gamma_f(\mathcal{E}')$. Moreover, this inequality is clearly strict if $g_f > 0$, as $\underline{\delta}_{\mathcal{E}} > \underline{\delta}_{\mathcal{E}'}$ for a positive measure of recommendations.

Similarly, for the back-end costs, we observe (see the proof of Lemma 10 in Appendix A.1) that

$$\Gamma_b(\mathcal{E}, Q) = \int_X \left[\int_0^{\underline{\delta}_{\mathcal{E}}(x)} g_b(x,z)\,\mathrm{d}z + \int_{\overline{\delta}_{\mathcal{E}}(x)}^{\bar{x}} g_b(x,z)\,\mathrm{d}z \right] \mathrm{d}Q(x) \tag{68}$$

and an analogous argument applies to show that $\Gamma_b(\mathcal{E}, Q) \geq \Gamma_b(\mathcal{E}', Q)$, for all Q. This suffices to show that the variant evidence structure $\mathcal{E}'$ implements the same outcomes and payments at lower front-end and back-end costs, completing the proof.

B.2. *Proof of Lemma 8*

(Only if) We begin by proving the necessity of the existence of a monotone tariff with respect to C. Suppose that ϕ is implementable. It follows that there exists (ξ, T) that supports ϕ. In particular, observe that (O) implies that $\phi(\theta) \in C(\xi(\theta))$ for all $\theta \in \Theta$. Next define $\hat{T} : X \to \bar{\mathbb{R}}$ as:

$$\hat{T}(x) = \inf_{y \in X} \{T(y) : x \in C(y)\}. \tag{69}$$

We next show that ϕ is also supported by $(\phi, \hat{T})$. By (O) of (ϕ, ξ, T), we have $u(\phi(\theta), \theta) \geq u(x, \theta)$ for all $x \in C(\phi(\theta)) \subseteq C(\xi(\theta))$ (by transitivity) and for all $\theta \in \Theta$, yielding (O) of $(\phi, \phi, \hat{T})$. By (IR) of (ϕ, ξ, T) and the definition of $\hat{T}$, we have

$$u(\phi(\theta), \theta) - \hat{T}(\phi(\theta)) \geq u(\phi(\theta), \theta) - T(\xi(\theta)) \geq 0, \tag{70}$$

for all $\theta \in \Theta$, yielding (IR) of $(\phi, \phi, \hat{T})$. Next, assume toward a contradiction that $(\phi, \phi, \hat{T})$ does not satisfy (IC), that is, there exist $\theta \in \Theta$ and $y \in X$ such that

$$\max_{x \in C(y)} u(x, \theta) - \hat{T}(y) > u(\phi(\theta), \theta) - \hat{T}(\phi(\theta)). \tag{71}$$

By the definition of $\hat{T}$, there exists a sequence $z_n \in X$ such that $y \in C(z_n)$ for all n and $T(z_n) \downarrow \hat{T}(y)$. Thus, there exists n large enough such that

$$\begin{aligned} \max_{x \in C(z_n)} u(x, \theta) - T(z_n) &\geq \max_{x \in C(y)} u(x, \theta) - T(z_n) > u(\phi(\theta), \theta) - \hat{T}(\phi(\theta)) \\ &\geq u(\phi(\theta), \theta) - T(\xi(\theta)) = \max_{x \in C(\xi(\theta))} u(x, \theta) - T(\xi(\theta)). \end{aligned} \tag{72}$$

The first inequality follows from $C(y) \subseteq C(z_n)$ since $y \in C(z_n)$. The second strict inequality follows from Equation 71 and the fact that $T(z_n) \downarrow \hat{T}(y)$. The third inequality follows from the construction of $\hat{T}$. The final equality follows as (ϕ, ξ, T) satisfies (O). However, the previous inequality yields a contradiction of (IC) of (ϕ, ξ, T), proving that $(\phi, \phi, \hat{T})$ satisfies (IC). This shows that $(\phi, \phi, \hat{T})$ is implementable, hence that Equation 32 holds and that $u(\phi(\theta), \theta) - \hat{T}(\phi(\theta)) \geq 0$ for all $\theta \in \Theta$.

Finally, we argue that $\hat{T}$ is monotone with respect to C. Fix $x, y \in X$ such that $y \in C(x)$. By Transitivity of C we have $\{\hat{x} \in X : x \in C(\hat{x})\} \subseteq \{\hat{x} \in X : y \in C(\hat{x})\}$, yielding that $\hat{T}(y) \leq \hat{T}(x)$, as desired.

(If) We now establish sufficiency. Suppose that there exists a tariff $T : X \to \bar{\mathbb{R}}$ that is monotone with respect to C and such that Equation 32 holds and $u(\phi(\theta), \theta) - T(\phi(\theta)) \geq 0$ for all $\theta \in \Theta$. We will show that (ϕ, ϕ, T) is implementable. (IR) is immediately satisfied. Next, we show that (IC) is satisfied. Suppose, toward a contradiction, that it were not. That is, there exist $\theta \in \Theta$, $y \in X$, and $x \in C(y)$ such that

$$u(x, \theta) - T(y) > \max_{\hat{x} \in C(\phi(\theta))} u(\hat{x}, \theta) - T(\phi(\theta)) \geq u(\phi(\theta), \theta) - T(\phi(\theta)). \tag{73}$$

But then, we have the following contradiction of monotonicity of T in C:

$$u(x, \theta) - T(y) > u(\phi(\theta), \theta) - T(\phi(\theta)) \geq u(x, \theta) - T(x) \implies T(x) > T(y), \tag{74}$$

where the second inequality uses the fact that $\phi(\theta)$ solves the program in Equation 32. Finally, we show that (O) is satisfied. Toward a contradiction, assume that there exists $\theta \in \Theta$ and $x \in C(\phi(\theta))$ such that $u(x, \theta) > u(\phi(\theta), \theta)$. However, by monotonicity of T in C, we know that

$T(\phi(\theta)) \geq T(x)$. Thus, $u(x,\theta) - T(x) > u(\phi(\theta),\theta) - T(\phi(\theta))$, yielding a contradiction to IC, which we just showed. This proves sufficiency.

Finally, the fact that any implementable final action function can be implemented as part of an allocation (ϕ, ϕ, T) follows by the construction in the necessity part of our proof.

B.3. *Proof of Proposition 1*

By Corollary 1, there always exists an optimal menu with finite actions, supported by a set of self-enforcing recommendations $\overline{D} = \{x_1, \ldots, x_K\}$ for some $K \in \mathbb{N}$ and such that $x_1 < x_2 < \ldots x_K$. Throughout, we adopt the notational convention that $x_0 = 0$ and $x_{K+1} = \bar{x}$.

We next derive the optimal allocation and payments. As shown in the last step of the proof of Lemma 5, for any given $\overline{D}$,

$$\phi^*(\theta) = \begin{cases} \overline{\phi}(\theta) = \min\{x \in \overline{D} : x \geq \phi^P(\theta)\}, & \text{if } H(\overline{\phi}(\theta),\theta) > H(\underline{\phi}(\theta),\theta), \\ \underline{\phi}(\theta) = \max\{x \in \overline{D} : x \leq \phi^P(\theta)\}, & \text{otherwise.} \end{cases} \tag{75}$$

We now specialize this using the structure of $\overline{D}$. As H is strictly single-crossing, $H(x_k,\theta) - H(x_{k-1},\theta) = 0$ has no solution if and only if (i) $H(x_k,0) - H(x_{k-1},0) > 0$ or (ii) $H(x_k,1) - H(x_{k-1},1) < 0$. As H is strictly quasi-concave, if $H(x_k,0) - H(x_{k-1},0) > 0$, then $H(\cdot,0)$ is strictly increasing at x_{k-1}, and therefore at all x_j for $j \leq k-1$. Thus, if $H(x_k,0) - H(x_{k-1},0) > 0$ holds for k, it holds for all $j \leq k$. Define $\underline{k} = \max\{k \in \{1,\ldots,K\} : H(x_k,0) - H(x_{k-1},0) > 0\}$, with the convention that $\underline{k} = 1$ if this set is empty. Similarly, if $H(x_k,1) - H(x_{k-1},1) < 0$, then $H(\cdot,1)$ is strictly decreasing at x_k. Thus, if $H(x_k,1) - H(x_{k-1},1) < 0$ holds for k, it holds for all $j \geq k$. Define $\overline{k} = \min\{k \in \{1,\ldots,K\} : H(x_k,1) - H(x_{k-1},1) < 0\}$, with the convention that $\overline{k} = K+1$ if this set is empty. As H is strictly single crossing, $\overline{k} > \underline{k}$. We now have that $H(x_k,\theta) - H(x_{k-1},\theta) = 0$ has a solution if and only if $k \in \{\underline{k}+1,\ldots,\overline{k}-1\}$ (if $\overline{k} = \underline{k}+1$, then this set is empty). For all $k \geq \overline{k}$, we have that $\hat{\theta}_k = 1$. For all $k \leq \underline{k}$, we have that $\hat{\theta}_k = 0$. For all $k \in \{\underline{k}+1,\ldots,\overline{k}-1\}$, we have that $\hat{\theta}_k$ is the unique solution to $H(x_k,\hat{\theta}_k) = H(x_{k-1},\hat{\theta}_k)$. As H is strictly quasi-concave, we know that $\phi^P(\hat{\theta}_k) \in (x_{k-1},x_k)$, which implies that $\underline{\phi}(\hat{\theta}_k) = x_{k-1}$ and $\overline{\phi}(\hat{\theta}_k) = x_k$. Thus, we have that $\phi^*(\theta) = x_k$ for all $\theta \in (\hat{\theta}_k, \hat{\theta}_{k+1}]$.

We now derive the tariff. In the strictly quasi-concave case, applying Equation 33 from Lemma 9, we have that:

$$\begin{aligned} T(x_k) &= u(x_k,\hat{\theta}_k) - \mathbb{I}[k \geq 2] \sum_{j=1}^{k-1} \int_{\hat{\theta}_j}^{\hat{\theta}_{j+1}} u_\theta(x_j,s)\,\mathrm{d}s \\ &= u(x_1,0) + \mathbb{I}[k \geq 2] \sum_{j=2}^{k} \left[u(x_j,\hat{\theta}_j) - u(x_{j-1},\hat{\theta}_j) \right], \end{aligned} \tag{76}$$

where the equality computes the integral and telescopes the summation. Observing that $x_1 = 0$ and $u(0,0) = 0$ gives the desired formula. The same formula trivially holds in the strictly quasi-convex case.

We next derive the necessary first-order condition. We first observe that, given a set of self-enforcing recommendations $\{x_1,\ldots,x_K\}$, the principal's payoff inclusive of back-end costs can be written as

$$\mathcal{H}(\{x_1,\ldots,x_K\}) = \sum_{k=1}^{K} \int_{\hat{\theta}_k}^{\hat{\theta}_{k+1}} H(x_k,\theta)\,\mathrm{d}F(\theta) \tag{77}$$

and, similarly, the front-end cost can be written as

$$\begin{aligned}\tilde{\Gamma}_f(\{x_1,\ldots,x_K\}) &= \int_X \int_{\overline{\delta}_{\overline{D}}(x)}^{\bar{x}} g_f(x,z)\,\mathrm{d}z\,\mathrm{d}x = \sum_{k=1}^{K-1} \int_{x_k}^{x_{k+1}} \int_{x_{k+1}}^{\bar{x}} g_f(x,z)\,\mathrm{d}z\,\mathrm{d}x \\ &= \sum_{k=1}^{K-1} \int_{x_k}^{x_{k+1}} \left(G_f(\bar{x},x) - G_f(x_{k+1},x)\right)\mathrm{d}x\,.\end{aligned} \tag{78}$$

A necessary condition for optimality is that payoffs cannot be locally improved by perturbing any action x_k, for $k \in \{2,\ldots,K-1\}$. That is,

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{H}(\{x_1,\ldots,x_k+\varepsilon,\ldots,x_K\})|_{\varepsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\tilde{\Gamma}_f(\{x_1,\ldots,x_k+\varepsilon,\ldots,x_K\})|_{\varepsilon=0}, \tag{79}$$

where we observe that both derivatives are well-defined. The left-hand side of this is

$$\begin{aligned}\int_{\hat{\theta}_k}^{\hat{\theta}_{k+1}} H_x(x_k^*,\theta)\,\mathrm{d}F(\theta) &= \int_{\hat{\theta}_k}^{\hat{\theta}_{k+1}} H_x(x_k^*,\theta)\,\mathrm{d}F(\theta) + \frac{\partial}{\partial x_k^*}\hat{\theta}_k \left(H(x_k^*,\hat{\theta}_k) - H(x_{k-1}^*,\hat{\theta}_k)\right) f(\hat{\theta}_k) \\ &\quad + \frac{\partial}{\partial x_k^*}\hat{\theta}_{k+1} \left(H(x_{k+1}^*,\hat{\theta}_{k+1}) - H(x_k^*,\hat{\theta}_{k+1})\right) f(\hat{\theta}_{k+1}),\end{aligned} \tag{80}$$

where, in the second equality, we use the fact that $H(x_k^*,\hat{\theta}_k) = H(x_{k-1}^*,\hat{\theta}_k)$ when $\hat{\theta}_k \in (0,1)$ and $\frac{\partial}{\partial x_k^*}\hat{\theta}_k = 0$ when $\hat{\theta}_k \in \{0,1\}$. Moreover, using the definition $H(x,\theta) = J(x,\theta) - \int_x^{\overline{x}} g_b(x,z)\,\mathrm{d}z$, we observe that:

$$H_x(x,\theta) = J_x(x,\theta) - \int_x^{\bar{x}} g_{b,x}(x,z)\,\mathrm{d}z + g_b(x,x), \tag{81}$$

where the latter two terms do not depend on θ. Thus,

$$\begin{aligned}\int_{\hat{\theta}_k}^{\hat{\theta}_{k+1}} H_x(x_k^*,\theta)\,\mathrm{d}F(\theta) &= \int_{\hat{\theta}_k}^{\hat{\theta}_{k+1}} J_x(x_k^*,\theta)\,\mathrm{d}F(\theta) + \\ &\quad \left(F(\hat{\theta}_{k+1}) - F(\hat{\theta}_k)\right)\left(g_b(x_k,x_k) - \int_{x_k}^{\bar{x}} g_{b,x}(x_k,z)\,\mathrm{d}z\right).\end{aligned} \tag{82}$$

The right-hand side of Equation 79 can be written as

$$\begin{aligned}&(G_f(\bar{x},x_k) - G_f(x_k,x_k)) - \int_{x_{k-1}}^{x_k} g_f(x,x_k)\,\mathrm{d}x - (G_f(\bar{x},x_k) - G_f(x_{k+1},x_k)) \\ &= G_f(x_{k+1},x_k) - G_f(x_k,x_k) - \int_{x_{k-1}}^{x_k} g_f(x,x_k)\,\mathrm{d}x\end{aligned} \tag{83}$$

We obtain the desired Equation 14 by plugging Equations 82 and 83 into Equation 79.

B.4. *Proof of Proposition 2*

We split the argument into three parts. We first map the problem to one that satisfies the general assumptions of Section 2. We next calculate the optimal contract in the transformed problem. We finally map the allocation and tariff back to the original problem.

Step 1: Transformation of the Problem We define $x = 1 - e \in X = [0,1]$ as the agent's shirking and $\theta = 1 - \vartheta \sim F = U[0,1]$ as their unproductivity. We define transformed preferences for the agent, which satisfy all of our maintained assumptions:

$$u(x,\theta) = \tilde{u}(1-x, 1-\theta) - \tilde{u}(1, 1-\theta) = (\alpha\theta + \beta)x - \beta\frac{x^2}{2}. \tag{84}$$

We define the transformed payoff for the principal,

$$\pi(x) = \tilde{\pi}(1-x) - \tilde{\pi}(1) = \eta(1-x) - \eta = -\eta x. \tag{85}$$

We next rewrite the front-end and back-end costs in terms of shirking. We first observe that an evidence structure over effort induces an evidence structure over shirking defined by $\mathcal{E}^x : X \to [0,1]$, where $X = [0,1]$ and, for each $x \in [0,1]$, $\mathcal{E}^x(x) = \mathcal{E}(1-x)$. Moreover, $\overline{\delta}_{\mathcal{E}^x}(x) = 1 - \underline{\delta}_{\mathcal{E}}(1-x)$ and $\underline{\delta}_{\mathcal{E}^x}(x) = 1 - \overline{\delta}_{\mathcal{E}}(1-x)$, for each $x \in [0,1]$. Thus, in some abuse of notation, we can write

$$\Gamma_f(\mathcal{E}) = \tilde{\Gamma}_f(\mathcal{E}^x) = \kappa_f \int_E \left(\underline{\delta}_{\mathcal{E}^x}(x) + (1 - \overline{\delta}_{\mathcal{E}^x}(x))\right) \mathrm{d}x\,. \tag{86}$$

An analogous argument applies to the back-end cost.

The (e,ϑ) problem satisfies the assumptions of Lemma 10 in the case of monotone *decreasing* preferences. In this case, the result implies that the principal problem given a set of self-enforcing efforts $\underline{D} \subseteq E$ is

$$\tilde{\mathcal{H}}(\underline{D}) := \max_{\tilde{\phi}} \int_0^1 \tilde{H}(\tilde{\phi}(\vartheta), \vartheta)\, \mathrm{d}\tilde{F}(\vartheta) \text{ s.t. } \tilde{\phi}(\vartheta') \geq \tilde{\phi}(\vartheta), \tilde{\phi}(\vartheta) \in \underline{D},\ \vartheta, \vartheta' \in [0,1] : \vartheta' \geq \vartheta \tag{87}$$

where the modified virtual surplus function is

$$\begin{aligned}\tilde{H}(e,\vartheta) &= \tilde{u}(e,\vartheta) + \tilde{\pi}(e) - \frac{1 - \tilde{F}(\vartheta)}{\tilde{f}(\vartheta)} \tilde{u}_\vartheta(e,\vartheta) - \kappa_b e \\ &= (2\alpha\theta + \beta - \eta + \kappa_b)x - \beta\frac{x^2}{2} + \left(\eta - \kappa_b - \frac{\beta}{2} - 2\alpha\theta\right).\end{aligned} \tag{88}$$

Using this substitution, we observe the following mapping to the (x,θ) problem:

$$\begin{aligned}\tilde{\mathcal{H}}(\underline{D}) - \left(\eta - \alpha - \kappa_b - \frac{\beta}{2}\right) = \mathcal{H}(\overline{D}) = \max_{\phi} \int_0^1 \left((2\alpha\theta + \beta - \eta + \kappa_b)\phi(\theta) - \beta\frac{\phi(\theta)^2}{2}\right) \mathrm{d}F(\theta) \\ \text{s.t.} \quad \phi(\theta') \geq \phi(\theta), \phi(\theta) \in \overline{D}, \quad \theta, \theta' \in [0,1] : \theta' \geq \theta,\end{aligned} \tag{89}$$

where $\overline{D} = \{(1-e) \in X : e \in \underline{D}\}$. Combining this with the earlier observation about front-end costs (Equation 86), we obtain that the original problem is equivalent to solving the transformed problem with modified surplus $H(x,\theta) = (2\alpha\theta + \beta - \eta + \kappa_b)x - \beta\frac{x^2}{2}$. Moreover, this function is strictly quasi-concave.

Corollary 1 implies that any optimal contractibility correspondence can be represented by a coarse set of self-enforcing shirking recommendations, $\overline{D} = \{x_1, \ldots, x_K\}$. In the transformed problem, these map to a set of self-enforcing effort recommendations $\{1 - x_1, \ldots, 1 - x_K\}$. To apply Proposition 1, we must also establish that (without loss of optimality) $x_1 = 0$ and $x_K = \overline{x} = 1$. *Excludability* guarantees that $e_1 = 0$ (zero effort, or full shirking) is perfectly contractible. An additional simple argument is required to establish that (without loss of optimality) we can restrict attention to the case in which $e_K = 1$ (full effort or zero shirking) is also contractible. Imagine it were not, and optimal contractibility were represented by an evidence structure $\mathcal{E}$. Then, construct a variant evidence structure $\mathcal{E}'$ that perturbs $\mathcal{E}$ such that $e = 1$ is self-enforcing. Since the evidence space $\Omega = [0,1]$ has the cardinality of the continuum, this is always possible by having $e = 1$ generate a unique (singleton) piece of evidence. We observe that $\Gamma_f(\mathcal{E}') = \Gamma_f(\mathcal{E})$ and $\Gamma_b(\mathcal{E}', Q) = \Gamma_b(\mathcal{E}, Q)$ for all Q, as both costs, when re-expressed with $(\underline{\delta}, \overline{\delta})$ as their arguments, are L_1-continuous. But under $\mathcal{E}'$ there are weakly more self-enforcing recommendations. Thus, $\mathcal{E}'$ obtains higher value. Hence, it is without loss of optimality to restrict attention to contractibility containing $e = 1$ ($x = 0$) as a self-enforcing recommendation. Together, this argument shows that we can apply Proposition 1 to the transformed problem.

Step 2: Optimal Contract in the Transformed Problem We next apply Proposition 1 to the transformed problem. The modified virtual surplus function follows Equation 18.

We first solve for the candidate optimal contract that solves the variational first-order condition in Proposition 1 for each K. This first-order condition for $k \in \{2, \ldots, K-1\}$ is

$$\int_{\hat{\theta}_k}^{\hat{\theta}_{k+1}} (2\alpha\theta + \beta(1 - x_k) - \eta + \kappa_b)\,\mathrm{d}\theta - \kappa_f(-2x_k + x_{k-1} + x_{k+1}) = 0, \tag{90}$$

where $\hat{\theta}_k = \frac{\beta}{4\alpha}(x_k + x_{k-1}) + \frac{\eta - \kappa_b - \beta}{2\alpha}$. Calculating the integral and evaluating at $\hat{\theta}_k$, we write

$$\kappa_f(-2x_k + x_{k-1} + x_{k+1}) = \frac{\beta^2}{16\alpha}(x_{k+1} - x_{k-1})(x_{k+1} + x_{k-1} - 2x_k). \tag{91}$$

This condition can be rearranged to obtain

$$(x_{k+1} + x_{k-1} - 2x_k)\left[\frac{\beta^2}{16\alpha}(x_{k+1} - x_{k-1}) - \kappa_f\right] = 0. \tag{92}$$

This equation has two solutions,

$$x_k = \frac{x_{k+1} + x_{k-1}}{2}, \qquad x_{k+1} = x_{k-1} + \Delta, \tag{93}$$

where $\Delta = \frac{16\alpha\kappa_f}{\beta^2}$. We now separately consider three cases.

We first consider case 1, in which all points are on the "uniform grid." From the boundary conditions, $x_1 = 0$ and $x_K = 1$. Thus, $x_k = \frac{x_{k+1} + x_{k-1}}{2} \implies x_k = \frac{k-1}{K-1}$. We can verify that this is a local maximum by checking that the Hessian is negative definite at this solution. We calculate that:

$$\begin{aligned}\frac{\partial^2(\mathcal{H}-\tilde{\Gamma}_f)}{\partial x_k^2} &= \mathrm{Hess}^{\mathcal{H}}_{k-1,k-1} = -\frac{\beta^2}{4\alpha(K-1)} + 2\kappa_f =: \Lambda, \\ \frac{\partial^2(\mathcal{H}-\tilde{\Gamma}_f)}{\partial x_k \partial x_{k+1}} &= \mathrm{Hess}^{\mathcal{H}}_{k,k-1} = \mathrm{Hess}^{\mathcal{H}}_{k-1,k} = \frac{\beta^2}{8\alpha(K-1)} - \kappa_f = -\frac{1}{2}\Lambda,\end{aligned} \tag{94}$$

where we note that row and column $k-1$ of $\mathrm{Hess}^{\mathcal{H}}$ corresponds to the variable x_k and in the first equality we define Λ. Thus, the Hessian is a tridiagonal Toeplitz matrix, which implies that the Eigenvalues are, by Theorem 2.2 of Kulkarni, Schmidt, and Tsui (1999), given by:

$$\lambda_k = \Lambda\left(1 - \cos\left(\frac{k-1}{K-1}\pi\right)\right) \tag{95}$$

for $k \in \{2, \ldots, K-1\}$. As $\cos\left(\frac{k-1}{K-1}\pi\right) \in (-1,1)$ for all such k, we have that $\mathrm{sgn}(\lambda_k) = \mathrm{sgn}(\Lambda)$. Thus, the Hessian is negative definite if and only if $K < \bar{K} = 1 + \frac{\beta^2}{8\alpha\kappa_f}$. We will later verify that this holds whenever K is set optimally, confirming the optimality of the uniform grid.

We next consider case 2, in which all points are on the "alternating grid." In this case, even points form a uniform grid with spacing $\Delta = \frac{16\alpha\kappa_f}{\beta^2}$ and the odd points form a uniform grid with spacing $\Delta = \frac{16\alpha\kappa_f}{\beta^2}$. When K is odd, given the boundary conditions that $x_1 = 0$ and $x_K = 1$, we have that this is possible only when $K = 1 + \frac{2}{\Delta}$, which is itself only possible when $\frac{\beta^2}{8\alpha\kappa_f}$ is an even integer. When K is even, the solution must be $x_k = \frac{k-1}{2}\Delta$ for k odd, and $x_k = 1 - \frac{K-k}{2}\Delta$ for k even. This is possible for any even K such that $\frac{2}{\Delta} < K < 2 + \frac{2}{\Delta}$.

We next show that the alternating grid (for odd or even K) is *not* a local maximum of the objective function. For a local maximum, a necessary condition is that the Hessian is negative semidefinite. We will show the existence of a vector $v \in \mathbb{R}^{K-2}$ such that $v \neq 0$ and $v'\mathrm{Hess}^{\mathcal{H}}v > 0$, which implies that $\mathrm{Hess}^{\mathcal{H}}$ is not negative semidefinite. To do this, we first calculate and simplify the second derivatives:

$$\begin{aligned}\frac{\partial^2(\mathcal{H}-\tilde{\Gamma}_f)}{\partial x_k^2} &= \mathrm{Hess}^{\mathcal{H}}_{k-1,k-1} = -\frac{\beta^2}{8\alpha}\Delta + 2\kappa_f = 0, \\ \frac{\partial^2(\mathcal{H}-\tilde{\Gamma}_f)}{\partial x_k \partial x_{k+1}} &= \mathrm{Hess}^{\mathcal{H}}_{k,k-1} = \mathrm{Hess}^{\mathcal{H}}_{k-1,k} = \frac{\beta^2}{8\alpha}(x_{k+1} - x_k) - \kappa_f.\end{aligned} \tag{96}$$

Using this, we define $v_k = z_{k-1} - z_k$, where z_k denotes the unit vector in some dimension k. This direction corresponds to increasing x_k and decreasing x_{k+1}. We calculate

$$v_k'\mathrm{Hess}^{\mathcal{H}}v_k = 2\left(\kappa_f - \frac{\beta^2}{8\alpha}(x_{k+1} - x_k)\right). \tag{97}$$

We now split the proof into two cases. First, consider the case in which $K > 4$. In this case, there must exist some x_k, x_{k+1} such that $x_{k+1} - x_k < \frac{\Delta}{2}$, since the grid is not uniform. Then $v_k'\mathrm{Hess}^{\mathcal{H}}v_k > 2\left(\kappa_f - \frac{\Delta\beta^2}{16\alpha}\right) = 0$ and, as desired, we have shown that the Hessian is not negative definite. Next, we consider the case in which $K = 4$. In this case, we take two candidate vectors. The first is $u = z_1 + z_2$, and we observe $u'\mathrm{Hess}^{\mathcal{H}}u = 2(\frac{\beta^2}{8\alpha}(x_3 - x_2) - \kappa_f)$. The second is $v_1 = z_1 - z_2$, and we observe $v_1'\mathrm{Hess}^{\mathcal{H}}v_1 = 2(\kappa_f - \frac{\beta^2}{8\alpha}(x_3 - x_2)) = -u'\mathrm{Hess}^{\mathcal{H}}u$. We finally note that, if $u'\mathrm{Hess}^{\mathcal{H}}u = v_1'\mathrm{Hess}^{\mathcal{H}}v_1 = 0$, then $x_3 - x_2 = \frac{8\alpha\kappa_f}{\beta^2} = \frac{\Delta}{2}$ and the alternating grid collapses to the uniform grid.

The final case to consider is one in which some points are separated as per the alternating grid and others are separated as per the uniform grid. However, in this case, when points are separated according to the alternating grid, the previous argument on the Hessian applies, contradicting optimality.

We next derive the profit and optimal tariff evaluated at the uniform-grid solution:

LEMMA 11: *The value to the monopolist of a K-item contract supported on a uniform grid can be written as $V(K) = \hat{\Pi}(K) - \hat{\Gamma}_f(K)$ where*

$$\hat{\Pi}(K) = \frac{\beta - \eta + \kappa_b + 2\alpha}{4\alpha}(2\alpha - \eta + \kappa_b) + \frac{\beta^2}{48\alpha}\frac{(2K-3)(2K-1)}{(K-1)^2} \quad \textit{and} \quad \hat{\Gamma}_f(K) = \frac{\kappa_f}{2}\frac{K-2}{K-1}. \tag{98}$$

Moreover, the optimal allocation is supported by the tariff

$$T^*(x_k) = \frac{1}{2}\frac{k-1}{K^*-1}\left(\beta + \eta - \kappa_b - \frac{\beta}{2}\frac{k-1}{K^*-1}\right). \tag{99}$$

PROOF: We observe that the principal's payoff, inclusive of back-end costs (but exclusive of front-end costs), can be written as

$$\mathcal{H}\left((x_k)_{k=1}^K\right) = \sum_{k=1}^{K}\int_{\hat{\theta}_k}^{\hat{\theta}_{k+1}} H(x_k, \theta)\,\mathrm{d}F(\theta). \tag{100}$$

Using this, we calculate

$$\begin{aligned}\hat{\Pi}(K) &= \sum_{k=1}^{K}\int_{\hat{\theta}_k}^{\hat{\theta}_{k+1}}\left((2\alpha\theta + \beta - \eta + \kappa_b)x_k - \beta\frac{x_k^2}{2}\right)\mathrm{d}\theta = \sum_{k=1}^{K}\left[\alpha x_k\theta^2 - x_k\left(\eta - \kappa_b - \beta + \frac{\beta}{2}x_k\right)\theta\right]_{\hat{\theta}_k}^{\hat{\theta}_{k+1}} \\ &= \frac{\beta}{2\alpha(K-1)}\sum_{k=2}^{K-1}\left(\alpha x_k(\hat{\theta}_{k+1} + \hat{\theta}_k) - x_k\left(\eta - \kappa_b - \beta + \frac{\beta}{2}x_k\right)\right) + (1 - \hat{\theta}_K)\left(\alpha(1 + \hat{\theta}_K) + \frac{\beta}{2} - \eta + \kappa_b\right),\end{aligned} \tag{101}$$

where, in the second line, we use that $\hat{\theta}_{k+1} - \hat{\theta}_k = \frac{\beta}{2\alpha(K-1)}$ for $k < K$ and that $\hat{\theta}_{K+1} = 1$ and $x_K = 1$. We simplify the summation term as

$$\sum_{k=2}^{K-1}\left(\alpha x_k(\hat{\theta}_{k+1} + \hat{\theta}_k) - x_k\left(\eta - \kappa_b - \beta + \frac{\beta}{2}x_k\right)\right) = \frac{\beta}{12(K-1)}(K-2)(2K-3), \tag{102}$$

where we use that $\hat{\theta}_k + \hat{\theta}_{k+1} = -\frac{\beta - \eta + \kappa_b}{\alpha} + \frac{\beta}{\alpha}x_k$. To simplify the second term, we observe that $\hat{\theta}_K = \left(\frac{\eta - \kappa_b - \beta}{2\alpha}\right) + \frac{\beta}{4\alpha}\left(1 + \frac{K-2}{K-1}\right)$ and therefore

$$(1 - \hat{\theta}_K)\left(\alpha(\hat{\theta}_K + 1) + \frac{\beta}{2} - \eta + \kappa_b\right) = \frac{\beta - \eta + \kappa_b + 2\alpha}{4\alpha}(2\alpha - \eta + \kappa_b) + \frac{\beta^2}{16\alpha(K-1)^2}(2K-3). \tag{103}$$

Putting this together and simplifying, we write:

$$\hat{\Pi}(K) = \frac{\beta - \eta + \kappa_b + 2\alpha}{4\alpha}(2\alpha - \eta + \kappa_b) + \frac{\beta^2}{48\alpha}\frac{(2K-3)(2K-1)}{(K-1)^2}. \tag{104}$$

We next show the desired representation of $\hat{\Gamma}$ by direct calculation:

$$\hat{\Gamma}(K) = \kappa_f \sum_{k=1}^{K-1} \frac{1}{K-1}\frac{k-1}{K-1} = \frac{\kappa_f}{(K-1)^2}\sum_{k=1}^{K-1}(k-1) = \frac{\kappa_f}{2}\frac{K-2}{K-1}. \tag{105}$$

We finally compute the tariff. Starting from Equation 15,

$$\begin{aligned} T^*(x_k) &= \mathbb{I}[k \geq 2] \sum_{j=2}^{k} \left[u(x_j, \hat{\theta}_j) - u(x_{j-1}, \hat{\theta}_j) \right] \\ &= \mathbb{I}[k \geq 2] \sum_{j=2}^{k} \left[(\alpha\hat{\theta}_j + \beta)(x_j - x_{j-1}) + \frac{\beta}{2}(x_{j-1}^2 - x_j^2) \right] \\ &= \frac{1}{2}\frac{k-1}{K^*-1}\left(\beta + \eta - \kappa_b - \frac{\beta}{2}\frac{k-1}{K^*-1}\right), \end{aligned} \tag{106}$$

where we substitute in the expressions for $\hat{\theta}_j$ and x_j, simplify at each step, and evaluate at $K = K^*$. *Q.E.D.*

To derive $\tilde{K}$, we take the first derivative of V:

$$V'(K) = \frac{\beta^2}{24\alpha(K-1)^3} - \frac{\kappa_f}{2(K-1)^2}. \tag{107}$$

We observe that $V'(K) > 0$ if and only if $K < \tilde{K} := \frac{\beta^2}{12\alpha\kappa_f} + 1$. We now prove that $|K^* - \tilde{K}| < 1$. If $K^* - \tilde{K} > 1$, then we know that $V(K^*-1) > V(K^*)$ as $V' < 0$ for all $K^*-1 < K < K^*$; this contradicts optimality. Similarly, if $\tilde{K} - K^* > 1$, we know that $V(K^*+1) > V(K^*)$ as $V' > 0$ for all $K^* < K < K^*+1$; this contradicts optimality. Recall that we needed to check if the Hessian was negative definite. This is true so long as $K^* \leq \bar{K}$. This holds whenever $\tilde{K} + 1 \leq \bar{K}$, which reduces to $\tilde{K} \geq 3$. It remains to check when $\tilde{K} \in (2,3)$ and $K^* = 3$. Direct calculation shows that indifference between $K = 2$ and $K = 3$ occurs when $\kappa_f = \frac{\beta^2}{16\alpha}$. At this point, $\tilde{K} = 7/3$. Thus, whenever $K^* > 2$ is strictly optimal (which is when $\kappa_f < \frac{\beta^2}{16\alpha}$), we have that $K^* < \bar{K}$. The comparative statics follow from standard monotone comparative statics arguments, after the observations that $V_{K\alpha} < 0$, $V_{K\beta} > 0$, $V_{K\eta} = 0$, and $V_{K\kappa_b} = 0$.

Step 3: Mapping Back to the Original Problem Since $e = 1 - x$, we observe that the optimal contract can be supported on $e_k = 1 - x_{K^*-k+1} = \frac{k-1}{K^*-1}$ with K^* self-enforcing recommendations. We next observe that, in the original problem, the IR constraint binds for type $\vartheta = 0$, who always (for any K^*) takes action $x = 1$ or $e = 0$ and receives direct payoff $\tilde{u}(0,0) = 0$. Therefore, $\tilde{T}(e_k) = T(x_{K^*-k+1}) - C$ where C solves $T(1) - C = 0$, and hence $C = \frac{1}{2}\left(\frac{\beta}{2} + \eta - \kappa_b\right)$.

Therefore, we calculate

$$\begin{aligned}\tilde{T}(e_k) &= \frac{1}{2}\left(1-\frac{k-1}{K^*-1}\right)\left(\beta+\eta-\kappa_b-\frac{\beta}{2}\left(1-\frac{k-1}{K^*-1}\right)\right)-\frac{1}{2}\left(\frac{\beta}{2}+\eta-\kappa_b\right)\\ &= -\frac{1}{2}\frac{k-1}{K^*-1}\left(\eta-\kappa_b+\frac{\beta}{2}\frac{k-1}{K^*-1}\right).\end{aligned} \tag{108}$$

We finally translate the transfers into wages by reversing the sign of the payments: $w(e_k) = -\tilde{T}(e_k)$ for all k. That is, a positive wage corresponds to a negative transfer from the agent (worker) to the principal (firm). This completes the proof.

B.5. *Proof of Corollary 2*

The first part of the statement follows immediately from the characterization of the optimal menu in Proposition 2. We now prove the second part. By assumption on $(\alpha,\beta,\eta,\kappa_b,\kappa_f)$, there exists a maximal neighborhood $\mathcal{B}$ of α such that, for all $\alpha' \in \mathcal{B}$, there exists a unique optimal K^* at $(\alpha',\beta,\eta,\kappa_b,\kappa_f)$. Let $\mathcal{A}$ be the set of $\alpha' \in \mathcal{B}$ such that $K^*(\alpha') = K^*(\alpha)$, where $K^*(\alpha')$ is the unique optimal K^* at parameter vector $(\alpha',\beta,\eta,\kappa_b,\kappa_f)$. We observe from Lemma 11 that $V = \hat{\Pi} - \hat{\Gamma}$ is submodular in α. This implies that K^* is monotone in α and therefore that $\mathcal{A} = \mathcal{B}$. By Proposition 2, we have that $\{w^*(e_k)\}_{k=1}^{K^*}$ is invariant to α conditional on K^*. Moreover, again by Proposition 2, $\{w^*(e_k)\}_{k=1}^{K^*}$ is different from $\{w^*(e_k)\}_{k=1}^{K}$ for any $K \neq K^*$ and, as $\mathcal{B}$ is the maximal neighborhood of α such that K^* is unique, whenever $\alpha' \notin \mathcal{B}$, there exists an optimal $K^{**} \neq K^*$.

C. REPRESENTATIONS OF EVIDENCE AND THEIR PROPERTIES

In this appendix, we derive several equivalent representations of evidence and establish their topological properties.

C.1. *Representing Evidence*

In this appendix, we prove Lemma 1. It follows as a consequence of the following, more general equivalence that we exploit in some of our proofs.

LEMMA 12: *The following four properties of a contractibility correspondence $C : X \rightrightarrows X$ are equivalent:*

1. *C is a regular contractibility correspondence, that is, $C = C_{\mathcal{E}}$ for some regular evidence structure $\mathcal{E} : X \rightrightarrows \Omega$ (see definition of $C_{\mathcal{E}}$ in Equation 22).*
2. *C satisfies (i) Reflexivity: for every $y \in X$, $y \in C(y)$; (ii) Excludability: $C(0) = \{0\}$; (iii) Transitivity: for every $x, y, z \in X$, if $x \in C(y)$ and $z \in C(x)$, then $z \in C(y)$; (iv) Monotonicity: for every $x, y \in X$, if $x \leq y$, then $C(x) \leq_{SSO} C(y)$, where $\leq_{SSO}$ denotes the strong set order; (v) Continuity: C is closed valued and lower hemicontinuous.*
3. *We have $C(y) = [\underline{\delta}(y), \overline{\delta}(y)]$ for all $y \in X$, where $\underline{\delta} : X \to X$ is an upper semi-continuous increasing function and $\overline{\delta} : X \to X$ is a lower semi-continuous increasing function such that: (i) $y \in [\underline{\delta}(y), \overline{\delta}(y)]$, (ii) $\underline{\delta}(x) = \underline{\delta}(y)$ for all $x \in [\underline{\delta}(y), y)$, (iii) $\overline{\delta}(x) = \overline{\delta}(y)$ for all $x \in (y, \overline{\delta}(y)]$, and (iv) $\overline{\delta}(0) = 0$.*
4. *There exist two closed sets $\underline{D} \subseteq X$ and $\overline{D} \subseteq X$ such that $0 \in \underline{D}$, $0, \overline{x} \in \overline{D}$ and for all $y \in X$:*

$$C(y) = \left[\max_{z \leq y : z \in \underline{D}} z,\ \min_{z \geq y : z \in \overline{D}} z\right]. \tag{109}$$

Moreover, given C, $\underline{D}$ and $\overline{D}$ are unique and given by $\underline{D} = \underline{\delta}(X)$ and $\overline{D} = \overline{\delta}(X)$.

PROOF: To prove this, we show that: 1 implies 2, 2 implies 3, 3 implies 4, and 4 implies 1.

1 implies 2 By Equation 22, it is immediate that any contractibility correspondence C that is induced by some evidentiary correspondence $\mathcal{E}$ is transitive, reflexive, closed-valued, and lower hemicontinuous. Observe that definitive evidence of exclusion implies that $C(0) = \{0\}$, yielding excludability of C. It remains only to show monotonicity of C. Fix $y' \geq y$, $x \in C(y)$, and $x' \in C(y')$. If $x' \geq x$, then $\max\{x, x'\} = x' \in C(y')$ and $\min\{x, x'\} = x \in C(y)$. Suppose now that $x' < x$. We now show that $\mathcal{E}(x) \subseteq \mathcal{E}(y')$ and $\mathcal{E}(x') \subseteq \mathcal{E}(y)$, which yields monotonicity of C by the fact that these claims imply that $\max\{x, x'\} = x \in C(y')$ and $\min\{x, x'\} = x' \in C(y)$. By definition we have that $\mathcal{E}(x) \subseteq \mathcal{E}(y)$, $\mathcal{E}(x') \subseteq \mathcal{E}(y')$, $\mathcal{E}(y') \geq_{SSO} \mathcal{E}(y)$, and $\mathcal{E}(x) \geq_{SSO} \mathcal{E}(x')$. Fix $\omega \in \mathcal{E}(x)$ and $\omega' \in \mathcal{E}(x')$. If $\omega \leq \omega'$, as $\mathcal{E}(x) \geq_{SSO} \mathcal{E}(x')$, we have that $\omega = \min\{\omega, \omega'\} \in \mathcal{E}(x')$. As $\mathcal{E}(x') \subseteq \mathcal{E}(y')$, this implies that $\omega \in \mathcal{E}(y')$. If $\omega > \omega'$, as $\omega \in \mathcal{E}(x) \subseteq \mathcal{E}(y)$ and $\omega' \in \mathcal{E}(x') \subseteq \mathcal{E}(y')$, we know that $\omega = \max\{\omega, \omega'\} \in \mathcal{E}(y')$ by the fact that $\mathcal{E}(y') \geq_{SSO} \mathcal{E}(y)$. These steps imply that $\mathcal{E}(x) \subseteq \mathcal{E}(y')$. Now fix $\omega \in \mathcal{E}(x')$ and $\omega' \in \mathcal{E}(x)$. If $\omega > \omega'$, we have that $\omega = \max\{\omega, \omega'\} \in \mathcal{E}(x)$ as $\mathcal{E}(x) \geq_{SSO} \mathcal{E}(x')$. As $\mathcal{E}(x) \subseteq \mathcal{E}(y)$, this implies that $\omega \in \mathcal{E}(y)$. If $\omega \leq \omega'$, as $\omega \in \mathcal{E}(x') \subseteq \mathcal{E}(y')$ and $\omega' \in \mathcal{E}(x) \subset \mathcal{E}(y)$, we have that $\omega = \min\{\omega, \omega'\} \in \mathcal{E}(y)$ as $\mathcal{E}(y') \geq_{SSO} \mathcal{E}(y)$. These steps establish that $\mathcal{E}(x') \subseteq \mathcal{E}(y)$.

2 implies 3 Let $C : X \rightrightarrows X$ be a regular contractibility correspondence and define $\underline{\delta}(y) = \min C(y)$ and $\overline{\delta}(y) = \max C(y)$ for all $y \in X$. By the fact that C is closed-valued, $\overline{\delta}$ and $\underline{\delta}$ exist. By monotonicity, we have that $\underline{\delta}$ and $\overline{\delta}$ are increasing functions. By reflexivity, we know that $y \geq \underline{\delta}(y)$ and $y \leq \overline{\delta}(y)$ for all y. Moreover, by Lemma 17.29 in Aliprantis and Border (2006), $\overline{\delta}$ is lower semi-continuous and $\underline{\delta}$ is upper semi-continuous.

We now show that $C(y) = [\underline{\delta}(y), \overline{\delta}(y)]$. Assume by contradiction there exists some $y \in X$ and $x \in [\underline{\delta}(y), \overline{\delta}(y)]$ such that $x \notin C(y)$. Consider first the case where $x < y$. By the definition of $\underline{\delta}$, $\underline{\delta}(y) \in C(y)$ and $\underline{\delta}(y) < x$. As $x < y$, by monotonicity, we have that $C(x) \leq_{SSO} C(y)$. Thus, as $x \in C(x)$ and $\underline{\delta}(y) \in C(y)$, we know that $\max\{x, \underline{\delta}(y)\} = x \in C(y)$. This is a contradiction. Consider now the case where $y < x$. Again, $\overline{\delta}(y) \in C(y)$ and $x < \overline{\delta}(y)$. By monotonicity, we have that $\min\{x, \overline{\delta}(y)\} = x \in C(y)$. This is a contradiction.

We next show parts (ii), (iii), and (iv). Fix $x, y \in X$ and assume that $x \in [\underline{\delta}(y), \overline{\delta}(y))$, which implies $x \in C(y)$. We start with part (ii), and mirror the argument for part (iii). Suppose $x < y$. As C is monotone, we know that $\underline{\delta}(x) \leq \underline{\delta}(y)$. Suppose by contradiction that $\underline{\delta}(x) < \underline{\delta}(y)$. But then, given the other properties of δ, for all $z \in (\underline{\delta}(x), \underline{\delta}(y))$ we would have that $z \in C(x)$ but $z \notin C(y)$, which contradicts transitivity. For part (iii), consider the same scenario but reversed. Suppose $x > y$. As C is monotone, we know that $\overline{\delta}(x) \geq \overline{\delta}(y)$. Imagine this held with a strict inequality. Then there would exist $z \in (\overline{\delta}(y), \overline{\delta}(x))$ such that $z \in C(x)$ and $z \notin C(y)$, while $x \in C(y)$. This violates transitivity. It is immediate that $\overline{\delta}(0) = 0$ by excludability as $C(0) = \{0\}$.

3 implies 4 We first observe the following for $\underline{\delta}, \overline{\delta}$ with the relevant properties: for all $\underline{z} \in \underline{\delta}(X)$ and $\overline{z} \in \overline{\delta}(X)$, it holds $\underline{\delta}(\underline{z}) = \underline{z}$ and $\overline{\delta}(\overline{z}) = \overline{z}$. To show this, let $\underline{z} = \underline{\delta}(x)$ for some $x \in X$. We have that $\underline{z} \in [\underline{\delta}(x), x]$. If $\underline{z} = x$, then we have that $\underline{\delta}(\underline{z}) = \underline{\delta}(x) = \underline{z}$. If $\underline{z} < x$, given property (ii), we must have $\underline{\delta}(\underline{z}) = \underline{\delta}(x) = \underline{z}$. The proof for $\overline{z} \in \overline{\delta}(X)$ is symmetric, using property (iii).

Define $\underline{D} = \underline{\delta}(X)$ and $\overline{D} = \overline{\delta}(X)$. First, observe that

$$\max_{z \leq x : z \in \underline{D}} z = \max_{z \leq x : z \in \underline{\delta}(X)} z \geq \underline{\delta}(x), \tag{110}$$

by construction. Let $\underline{z} = \max_{z \leq x: z \in \underline{D}} z$ and assume by contradiction that $\underline{z} > \underline{\delta}(x)$. If $\underline{z} = x$, then $x \in \underline{\delta}(X)$ and by the earlier argument we have that $x = \underline{\delta}(x) < \underline{z}$, yielding a contradiction. If instead $\underline{z} < x$, then by the earlier argument and the property (ii) of $\underline{\delta}$, we have $\underline{z} = \underline{\delta}(\underline{z}) = \underline{\delta}(x)$, yielding a contradiction. With this, we conclude that $\underline{z} = \underline{\delta}(x)$. With symmetric steps, we can show that $\min_{z \geq x: z \in \overline{D}} z = \overline{\delta}(x)$. Next, observe that necessarily we have $\underline{\delta}(0) = 0$, $\overline{\delta}(\overline{x}) = \overline{x}$, and $\overline{\delta}(0) = 0$ proving that $0 \in \underline{D}$ and $0, \overline{x} \in \overline{D}$. Finally, we need to show that $\underline{D}$ and $\overline{D}$ are closed. Take a sequence $z_n \in \underline{D}$ such that $z_n \to z$. Given that X is closed, we have that $z \in X$ and therefore $\underline{\delta}(z) \leq z$. Given that every z_n is in $\underline{D}$, the property defined above implies that $\underline{\delta}(z_n) = z_n$ for all n. Given that $\underline{\delta}$ is upper semi-continuous, it follows that $z = \lim_{n \to \infty} z_n = \lim_{n \to \infty} \underline{\delta}(z_n) \leq \underline{\delta}(z)$. This implies that $z = \underline{\delta}(z)$ (as $z \geq \underline{\delta}(z)$) and therefore that $z \in \underline{D}$. This shows that $\underline{D}$ is closed. A symmetric argument shows that $\overline{D}$ is closed.

4 implies 1 We show this by construction. For the argument, we take $\Omega = X$; as Ω has the cardinality of the continuum, this is without loss of generality, as the construction below can be composed with an isomorphism from X to Ω. We define

$$\mathcal{E}(x) = \left[\max_{z \leq x: z \in \underline{D}} z, \min_{z \geq x: z \in \overline{D}} z \right] =: \left[\underline{\delta}(x), \overline{\delta}(x) \right], \tag{111}$$

where, for convenience, we define $\underline{\delta}, \overline{\delta}$ in the second equality. We observe immediately that these functions have the three properties from part 3 of the statement. It is immediate to see that both these functions are monotone increasing, such that $\underline{\delta}(x) \leq x \leq \overline{\delta}(x)$, and respectively upper semi-continuous and lower semi-continuous by Lemma 17.30 in Aliprantis and Border (2006). To see the remaining properties, observe that the correspondences $x \rightrightarrows \{z \in \underline{D} : z \leq x\}$ and $x \rightrightarrows \left\{z \in \overline{D} : z \geq x\right\}$ are both upper hemicontinuous. Next, assume that $y \in [\underline{\delta}(x), x)$ and let $z = \underline{\delta}(x)$. We have $\underline{\delta}(y) \leq z$ by monotonicity. Moreover, by assumption $z \leq y$ and $z \in \underline{D}$, so that $z \leq \underline{\delta}(y)$ by definition. We then must have $z = \underline{\delta}(y)$. Symmetrically, assume that $y \in (x, \overline{\delta}(x)]$ and let $z = \overline{\delta}(x)$. We have $\overline{\delta}(y) \geq z$ by monotonicity. Moreover, by assumption $z \geq y$ and $z \in \overline{D}$, so that $z \geq \overline{\delta}(y)$ by definition. We then must have $z = \overline{\delta}(y)$. Finally, as $0 \in \overline{D}$, we have that $\overline{\delta}(0) = 0$.

We now show that the induced evidence structure satisfies definitive evidence of exclusion, monotonicity, and the continuity property. Definitive evidence of exclusion follows from $\overline{\delta}(0) = 0$, and $\overline{\delta}(x) \geq x > 0$ for any $x \neq 0$. Thus, $\mathcal{E}(0) = \{0\}$ and $\mathcal{E}(x) \nsubseteq \mathcal{E}(0)$ for any $x \neq 0$.

We next show monotonicity. To remind, this property implies that, for all $x, x' \in X$ and $e, e' \in \Omega$, if $e' \geq e$ and $x' \geq x$, then

$$\mathbb{I}[e' \in \mathcal{E}(x')]\mathbb{I}[e \in \mathcal{E}(x)] \geq \mathbb{I}[e' \in \mathcal{E}(x)]\mathbb{I}[e \in \mathcal{E}(x')]. \tag{112}$$

This amounts to showing that the intersection of conditions on the right-hand side implies the intersection of conditions on the left-hand side. If both right-hand side conditions hold, then $\underline{\delta}(x) \leq e' \leq \overline{\delta}(x)$ and $\underline{\delta}(x') \leq e \leq \overline{\delta}(x')$. Using the monotonicity of $\underline{\delta}, \overline{\delta}$, we have that $e \leq e' \leq \overline{\delta}(x) \leq \overline{\delta}(x')$ and $\underline{\delta}(x) \leq \underline{\delta}(x') \leq e \leq e'$. Thus, $\underline{\delta}(x) \leq e \leq \overline{\delta}(x)$ and $\underline{\delta}(x') \leq e' \leq \overline{\delta}(x')$. By implication, as desired, $e' \in \mathcal{E}(x')$ and $e \in \mathcal{E}(x)$.

We next show continuity of $\mathcal{E}$. Fix $x, y \in X$. First, consider a sequence $x_n \to x$ with $\mathcal{E}(x_n) \subseteq \mathcal{E}(y)$ for all n, that is, $\underline{\delta}(y) \leq \underline{\delta}(x_n) \leq \overline{\delta}(x_n) \leq \overline{\delta}(y)$ for all n. The continuity properties of $\underline{\delta}$ and $\overline{\delta}$ imply that $\mathcal{E}(x) \subseteq \mathcal{E}(y)$. Second, assume $\mathcal{E}(x) \subseteq \mathcal{E}(y)$ and fix a sequence $y_n \to y$. Define $x_n = \min\left\{\max\left\{x, \underline{\delta}(y_n)\right\}, \overline{\delta}(y_n)\right\}$. By construction, we have $\mathcal{E}(x_n) \subseteq \mathcal{E}(y_n)$ for all n. By the semicontinuity properties of $\underline{\delta}$ and $\overline{\delta}$, we have $\limsup_n \underline{\delta}(y_n) \leq \underline{\delta}(y) \leq x$ and $\liminf_n \overline{\delta}(y_n) \geq \overline{\delta}(y) \geq x$. With this, we have that $x_n \to x$ as desired. *Q.E.D.*

C.2. *The Topology of Sets of Self-Enforcing Recommendations*

Let $\underline{\Delta} \times \overline{\Delta}$ denote the space of pairs of functions $(\underline{\delta}, \overline{\delta})$ that have all the properties in Lemma 1. Recall that we endow this set with the relative topology induced by the product L_1 topology over pairs of integrable functions. Also, recall that $\underline{\mathcal{D}}$ denotes the collection of closed subsets of X that contain 0 and $\overline{\mathcal{D}}$ denotes the collection of closed subsets of X that contain 0 and $\overline{x}$. Let $\underline{\mathcal{D}} \times \overline{\mathcal{D}}$ denote the set of pairs of self-enforcing recommendations sets, $(\underline{D}, \overline{D})$. Recall that we have endowed this space with the product topology induced by the Hausdorff topology on each collection of sets.

LEMMA 13: *The set $\underline{\mathcal{D}} \times \overline{\mathcal{D}}$ is compact.*

PROOF: We show that each of $\underline{\mathcal{D}}$ and $\overline{\mathcal{D}}$ is compact in the Hausdorff topology. Specifically, we explicitly establish the compactness of $\overline{\mathcal{D}}$ and observe that an entirely symmetric argument applies to establish the compactness of $\underline{\mathcal{D}}$. With this, the compactness of the product space $\underline{\mathcal{D}} \times \overline{\mathcal{D}}$ follows from Tychonoff's theorem.

Observe that $\overline{\mathcal{D}} \subset \mathfrak{F}$, where $\mathfrak{F}$ is the collection of nonempty closed sets of X. Theorem 3.85 in Aliprantis and Border (2006) establishes that $\mathfrak{F}$ is compact in the Hausdorff topology. Fix a sequence $\{\overline{D}_n\}_{n\in\mathbb{N}} \subseteq \overline{\mathcal{D}}$. By the compactness of $\mathfrak{F}$, this sequence must have a subsequence $\overline{D}_{n_k}$ converging to some $\overline{D} \in \mathfrak{F}$ in the Hausdorff topology. Furthermore, we have that $0, \overline{x} \in \overline{D}_{n_k}$ for all k. Hence, by Hausdorff convergence of $\{\overline{D}_{n_k}\}_{k\in\mathbb{N}}$, we have that $0, \overline{x} \in \overline{D}$. Since $\overline{D}$ is a closed subset of X that contains 0 and $\overline{x}$, we have $\overline{D} \in \overline{\mathcal{D}}$. Finally, because the initial sequence was arbitrarily chosen, this implies that $\overline{\mathcal{D}}$ is compact. *Q.E.D.*

Lemma 1 implies that $\underline{\Delta} \times \overline{\Delta}$ and $\underline{\mathcal{D}} \times \overline{\mathcal{D}}$ are isomorphic via the maps:

$$\begin{aligned} (\underline{\delta}, \overline{\delta}) &\mapsto (\underline{D}_{\underline{\delta}}, \overline{D}_{\overline{\delta}}) := (\underline{\delta}(X), \overline{\delta}(X)), \\ (\underline{D}, \overline{D}) &\mapsto (\underline{\delta}_{\underline{D}}(x), \overline{\delta}_{\overline{D}}(x)) := \left(\max\{z \in \underline{D} : z \leq x\}, \min\{z \in \overline{D} : z \geq x\}\right). \end{aligned} \tag{113}$$

We next prove that Hausdorff's convergence of sets of self-enforcing recommendations implies L_1-convergence of the corresponding envelope functions to the envelope functions induced by the limit sets.

LEMMA 14: *Fix two sequences $\{(\underline{D}_n, \overline{D}_n)\}_{n\in\mathbb{N}} \subseteq \underline{\mathcal{D}} \times \overline{\mathcal{D}}$ and $\{(\underline{\delta}_n, \overline{\delta}_n)\}_{n\in\mathbb{N}} \subseteq \underline{\Delta} \times \overline{\Delta}$ such that $(\underline{\delta}_n, \overline{\delta}_n) = (\underline{\delta}_{\underline{D}_n}, \overline{\delta}_{\overline{D}_n})$ for all $n \in \mathbb{N}$. If $(\underline{D}_n, \overline{D}_n) \to (\underline{D}, \overline{D})$, then $(\underline{\delta}_n, \overline{\delta}_n) \to (\underline{\delta}_{\underline{D}}, \overline{\delta}_{\overline{D}})$.*

PROOF: Fix two sequences as in the statement and assume that $(\underline{D}_n, \overline{D}_n) \to (\underline{D}, \overline{D})$ in the product Hausdorff topology. We only show that $\overline{\delta}_n \to \overline{\delta}_{\overline{D}}$ in L_1, as $\underline{\delta}_n \to \underline{\delta}_{\underline{D}}$ follows from an entirely symmetric argument. Together, these two facts imply convergence in the L_1 product topology.

For notational simplicity, denote $\overline{\delta}_{\overline{D}} = \overline{\delta}$ and let $X_{\overline{\delta}} \subseteq X$ denote the collection of points at which $\overline{\delta}$ is continuous. Because $\overline{\delta}$ is non-decreasing, it follows that $X \setminus X_{\overline{\delta}}$ is at most countable. We next show that $\overline{\delta}_n(x) \to \overline{\delta}(x)$ for all $x \in X_{\overline{\delta}}$. Before showing this, we observe that the previous claim concludes the argument because

$$\lim_{n\to\infty} \int_X |\overline{\delta}_n(x) - \overline{\delta}(x)|\, \mathrm{d}x = \lim_{n\to\infty} \int_{X_{\overline{\delta}}} |\overline{\delta}_n(x) - \overline{\delta}(x)|\, \mathrm{d}x = \int_{X_{\overline{\delta}}} \lim_{n\to\infty} |\overline{\delta}_n(x) - \overline{\delta}(x)|\, \mathrm{d}x = 0, \tag{114}$$

where the first equality follows from the fact that $X_{\overline{\delta}}$ has full measure, the second follows from the dominated convergence theorem, and the last follows from the claim.

Next, fix $x \in X_{\overline{\delta}}$. Clearly, if $x = \overline{x}$, then $\overline{\delta}_n(x) \to \overline{\delta}(x)$ since they are all equal to $\overline{x}$. Thus, we next always assume that $x < \overline{x}$. We split the rest of the proof of the claim into two parts depending on whether x is inside or outside $\overline{D}$.

1 Assume that $x \in X_{\overline{\delta}} \setminus \overline{D}$ and define $\overline{D}_n(x) = \overline{D}_n \cap [x, \overline{x}]$ for all n as well as $\overline{D}(x) = \overline{D} \cap [x, \overline{x}]$. We first show that $\overline{D}_n(x) \to \overline{D}(x)$ in the Hausdorff topology. Let $Li(\overline{D}_n(x))$ and $Ls(\overline{D}_n(x))$ respectively denote the topological limit inferior and the topological limit superior of the sequence $\{\overline{D}_n(x)\}_{n\in\mathbb{N}}$.[22] Next, fix $z \in \overline{D}(x)$ and observe that $z > x$ because $x \in X_{\overline{\delta}} \setminus \overline{D}$. Because $z \in \overline{D}$ and $\overline{D}_n \to \overline{D}$, it follows that there exists a sequence $z_n \in \overline{D}_n$ such that $z_n \to z$. In particular, we must have $z_n > x$ for all n large enough, yielding that $z_n \in \overline{D}_n(x)$ for all n large enough, hence that $z \in Li(\overline{D}_n(x))$. Because z was arbitrarily chosen, this implies that $\overline{D}(x) \subseteq Li(\overline{D}_n(x))$. Next, fix $z \in Ls(\overline{D}_n(x))$. By definition of Ls, there exists a sequence $\{z_k\}_{k\in\mathbb{N}}$ such that $z_k \to z$ and $z_k \in \overline{D}_{n_k}(x)$ along a subsequence parametrized by k. Because $\overline{D}_{n_k} \to \overline{D}$ in the Hausdorff topology by assumption and $z_k \in \overline{D}_{n_k}$ for all k, it follows that $z \in \overline{D}$. Similarly, because $z_k \in [x, \overline{x}]$ for all k, it follows that $z \in [x, \overline{x}]$, yielding that $z \in \overline{D}(x)$. Because z was arbitrarily chosen, this implies that $Ls(\overline{D}_n(x)) \subseteq \overline{D}(x)$ and overall that $\overline{D}(x) = Li(\overline{D}_n(x)) = Ls(\overline{D}_n(x))$. Theorem 3.93 in Aliprantis and Border (2006) then yields that $\overline{D}_n(x) \to \overline{D}(x)$ in the Hausdorff topology. Finally, because $\overline{\delta}_n(x) = \min \overline{D}_n(x)$ for all n and $\overline{\delta}(x) = \min \overline{D}(x)$, Theorem 17.31 in Aliprantis and Border (2006) implies that $\overline{\delta}_n(x) \to \overline{\delta}(x)$.

2 Assume that $x \in X_{\overline{\delta}} \cap \overline{D}$ and define $\overline{D}_n(x)$ for all n and $\overline{D}(x)$ as above. Also, let $\operatorname{int} \overline{D}$ and $\partial \overline{D}$ respectively denote the interior and the boundary points of $\overline{D}$. First, observe that $\overline{\delta}(x) = x$ because $x \in \overline{D}$. We next show that, for every $\varepsilon > 0$, there exists a point $x^\varepsilon \in \overline{D}(x)$ such that $0 < |x^\varepsilon - x| \leq \varepsilon$. We split the proof of this claim into two cases:

a. If $x \in \operatorname{int} \overline{D}$, then the claim is immediately true because $\operatorname{int} \overline{D}$ is open.
b. Assume now that $x \in \partial \overline{D}$ and, by contradiction, that there exists $\varepsilon > 0$ such that for all $z \in \overline{D}(x)$ we have either $|z - x| = 0$ or $|z - x| > \varepsilon$. Because $x < \overline{x}$, there must be $z \in \overline{D}(x)$ with $|z - x| > 0$, hence for all such z we must have $|z - x| > \varepsilon$. In turn, this implies that $\overline{D} \cap [x, x + \varepsilon] = \{x\}$, and hence that $\overline{\delta}(y) > x + \varepsilon$ for all $y \in (x, x + \varepsilon)$. With this, we have that $\overline{\delta}$ is discontinuous at x, yielding a contradiction.

Next, fix $\varepsilon > 0$ and $x^\varepsilon \in \overline{D}(x)$ as above. Because $x^\varepsilon \in \overline{D}$, Hausdorff convergence implies that there exists a sequence $x_n^\varepsilon \in \overline{D}_n$ such that $x_n^\varepsilon \to x^\varepsilon$. Moreover, because $x^\varepsilon > x$, for n large enough we must have $x_n^\varepsilon > x$, hence that $x_n^\varepsilon \in \overline{D}_n$. Therefore, for all n large enough, we have

$$x \leq \overline{\delta}_n(x) = \min \overline{D}_n(x) \leq x_n^\varepsilon. \tag{115}$$

Passing to the limits, we have

$$x \leq \liminf_n \overline{\delta}_n(x) \leq \limsup_n \overline{\delta}_n(x) \leq x^\varepsilon. \tag{116}$$

By taking $\varepsilon \to 0$ we conclude that $\lim_n \overline{\delta}_n(x) = x = \overline{\delta}(x)$, as desired. *Q.E.D.*

Before stating the main result of this section, observe that each cost function $\Gamma_\mathcal{E}$ induces cost functions in the spaces of envelope functions and sets of self-enforcing recommendations (by

[22] See for example Definition 3.80 in Aliprantis and Border (2006).

Lemma 1). In particular, we write these induced costs as $\Gamma_\Delta : \underline{\Delta} \times \overline{\Delta} \to [0,\infty]$ defined over the envelope functions of regular contractibility correspondences and $\Gamma_{\mathcal{D}} : \underline{\mathcal{D}} \times \overline{\mathcal{D}} \to [0,\infty]$ defined over the sets of self-enforcing recommendations of regular contractibility correspondences. Of course these induced costs are related by the identity $\Gamma_{\mathcal{D}}(\underline{D}, \overline{D}) = \Gamma_\Delta(\underline{\delta}_{\underline{D}}, \overline{\delta}_{\overline{D}})$.

PROPOSITION 3: *If a cost function Γ_Δ is lower semi-continuous (resp. continuous) in the product L_1 topology, then $\Gamma_{\mathcal{D}}$ is lower semi-continuous (resp. continuous) in the product Hausdorff topology.*

PROOF: The result immediately follows from Lemma 14. *Q.E.D.*

This proposition is useful in our analysis as it is simple to verify that front-end costs are continuous in the product L_1 topology when written in terms of envelopes of contractibility correspondences (as they are simply integrals of these functions). However, it is not simple to directly verify Hausdorff continuity of the cost in the space of self-enforcing recommendations. This result provides that conclusion, which is an essential step in our existence proof.

D. FLEXIBLE USE OF EVIDENCE

Our baseline analysis imposes a seeming restriction on the principal's use of evidence: payments depend only on the promised action and the binary event of whether they can prove that the agent violated the contract. In this appendix, we allow the principal to freely contract on the promised action and the evidence that they present to the arbitrator. We establish that, for any equilibrium under a mechanism that uses evidence flexibly, the principal can design a breach mechanism under which an equilibrium exists that yields the same expected payoff. Thus, it is without loss of optimality for the principal to employ breach mechanisms, justifying our focus on them.

A flexible mechanism $P : X \times 2^\Omega \to \mathbb{R}$ specifies payments from the agent to the principal $P(y, E)$ when the agent promises to do $y \in X$ and the principal presents evidence $E \subseteq \Omega$ to the arbitrator. The principal can present any subset E of the evidence generated by the agent's action $\mathcal{E}(x)$ to the arbitrator. Such a mechanism allows, for example, the principal to punish different violations of the contract in different ways (or not at all). The timing is as in our baseline model, but here we are explicit that after evidence has been generated, the principal elects which subset to exhibit to the arbitrator, who then executes payments in accordance with the flexible mechanism.

We also allow for limited liability in the form of a cap $\bar{M} \in \bar{\mathbb{R}}_{++}$ on the possible transfer from the agent to the principal: $P(y, E) \le \bar{M}$ for all $y \in X$ and $E \subseteq \Omega$. If a mechanism P respects this constraint, we say that it respects limited liability; in the limit case where $\bar{M} = \infty$, this is trivially satisfied by any mechanism. We describe an equilibrium (a pure SPNE) as a joint distribution $W \in \Delta(\Theta \times X \times X)$ over tuples (θ, y, x) of types, promised actions, and final actions.

Our analysis in the main text implicitly restricts to a simpler class of mechanisms:

DEFINITION 4—Breach Mechanism: *Given a regular evidence structure $\mathcal{E}$, a flexible mechanism $P : X \times 2^\Omega \to \mathbb{R}$ is a breach mechanism for $\mathcal{E}$ if there exists a tariff $T : X \to \mathbb{R}$ and a punishment $M \in \mathbb{R}_{++}$ such that:*

$$P(y, E) = \begin{cases} T(y), & E \cap \mathcal{E}(y)^c = \emptyset, \\ M, & \textit{otherwise}. \end{cases} \tag{117}$$

Under a breach mechanism, payments $T(y)$ are provided when the agent promises to do $y \in X$ so long as the principal cannot prove that the agent did not do y, *i.e.,* $E \cap \mathcal{E}(y)^c = \emptyset$. When the principal can prove that the agent did not do y, they levy a punishment M. A breach mechanism respects limited liability if $\max\left\{\sup_{y\in X} T(y), M\right\} \leq \bar{M}$.

The following result establishes that breach mechanisms are without loss of optimality, both for our baseline analysis and if limited liability is imposed:

PROPOSITION 4: *Fix some limited liability cap $\bar{M}$. For every regular evidence structure $\mathcal{E}$, contract P that respects limited liability $\bar{M}$, and equilibrium action distribution W given $(\mathcal{E}, P)$, there exists a breach mechanism $\hat{P}$ for $\mathcal{E}$ that respects limited liability $\bar{M}$ and an equilibrium action distribution $\hat{W}$ given $(\mathcal{E}, \hat{P})$ such that the principal's expected payoff from $(\mathcal{E}, P, W)$ equals that of $(\mathcal{E}, \hat{P}, \hat{W})$.*

PROOF: Fix an equilibrium W under evidence structure $\mathcal{E}$ and contract P. Let $\phi : \Theta \to X$ and $\xi : \Theta \to X$ denote the equilibrium mappings from types to final actions and promises, respectively. We also define the value of the principal optimally exhibiting evidence as:

$$\mathcal{T}(x,y) = \max_{E\subseteq\mathcal{E}(x)} P(y,E), \tag{118}$$

which must have a solution by the hypothesis that we are in an equilibrium. Moreover, let $V(\theta) = u(\phi(\theta),\theta) - \mathcal{T}(\phi(\theta),\xi(\theta))$ denote type θ's equilibrium payoff under P.

We first set the punishment for the breach mechanism M. If $\bar{M} < \infty$, set $M = \bar{M}$. If $\bar{M} = \infty$, choose any finite $M > \max_{x\in X,\theta\in\Theta} u(x,\theta)$. In either case,

$$u(x,\theta) - M \leq V(\theta) \quad \text{for all } x \in X, \theta \in \Theta. \tag{119}$$

When $\bar{M} < \infty$, this follows because $\mathcal{T}(x,y) \leq \bar{M} = M$ for all (x,y) and equilibrium under P implies $V(\theta) \geq u(x,\theta) - \mathcal{T}(x,y)$ for every deviation (y,x).

We fix a map $\zeta : X \to X$ such that:

$$\zeta(z) \in \arg\min_{y\in X} \mathcal{T}(z,y) \quad \forall z \in \phi(\Theta). \tag{120}$$

To see that such a map exists, fix $z \in \phi(\Theta)$ and choose θ_z such that $\phi(\theta_z) = z$. If there existed $y \in X$ such that $\mathcal{T}(z,y) < \mathcal{T}(z,\xi(\theta_z))$, then type θ_z could promise y and still take action z, contradicting equilibrium under P. Hence $\xi(\theta_z) \in \arg\min_{y\in X} \mathcal{T}(z,y)$, and the argmin is nonempty. In particular, for every type θ,

$$\mathcal{T}(\phi(\theta),\zeta(\phi(\theta))) = \mathcal{T}(\phi(\theta),\xi(\theta)). \tag{121}$$

Putting this together, we define the map $T : X \to \mathbb{R}$ such that:

$$T(z) = \begin{cases} \mathcal{T}(z,\zeta(z)), & z \in \phi(\Theta), \\ M, & \text{else.} \end{cases} \tag{122}$$

By construction, $T(z) \leq M$ for all $z \in X$. We take $\hat{P}$ as the breach mechanism induced by tariff T and penalty M.

Any agent can secure their equilibrium payoff $V(\theta)$ under P also under $\hat{P}$ by taking the same final action and promising to take this action. To see this, note that if any agent $\theta \in \Theta$ promises $\phi(\theta)$ and takes action $\phi(\theta)$ under $\hat{P}$, then they pay the principal

$$T(\phi(\theta)) = \mathcal{T}(\phi(\theta),\zeta(\phi(\theta))) = \mathcal{T}(\phi(\theta),\xi(\theta)), \tag{123}$$

where the last equality follows from Equation 121. Thus, it remains to show that the agent can do no better under $\hat{P}$ than under P. We now define:

$$\widehat{\mathcal{T}}(x,y) = \max_{E \subseteq \mathcal{E}(x)} \hat{P}(y,E), \tag{124}$$

for the induced transfer under $\hat{P}$. Toward a contradiction, suppose that there exists $x' \in X$, $y' \in X$ and a type $\theta \in \Theta$ such that:

$$u(x',\theta) - \widehat{\mathcal{T}}(x',y') > V(\theta). \tag{125}$$

There are two possible cases. First, suppose that $y' \in \phi(\Theta)$ and y' is such that $\mathcal{E}(x') \cap \mathcal{E}(y')^c = \emptyset$. In this case, we have that $\mathcal{E}(x') \subseteq \mathcal{E}(y')$. Thus, we have that:

$$u(x',\theta) - \widehat{\mathcal{T}}(x',y') = u(x',\theta) - T(y') = u(x',\theta) - \mathcal{T}(y',\zeta(y')). \tag{126}$$

Because $\mathcal{E}(x') \subseteq \mathcal{E}(y')$, for all $z \in X$,

$$\mathcal{T}(x',z) = \max_{E \subseteq \mathcal{E}(x')} P(z,E) \leq \max_{E \subseteq \mathcal{E}(y')} P(z,E) = \mathcal{T}(y',z). \tag{127}$$

In particular, taking $z = \zeta(y')$ and combining the previous two equations, we have that:

$$u(x',\theta) - \widehat{\mathcal{T}}(x',y') \leq u(x',\theta) - \mathcal{T}(x',\zeta(y')) \leq V(\theta), \tag{128}$$

where the last inequality follows from equilibrium under P, since under the original flexible mechanism type θ could have promised $\zeta(y')$ and taken action x'. This yields a contradiction.

Second, suppose that either $y' \notin \phi(\Theta)$ or y' is such that $\mathcal{E}(x') \cap \mathcal{E}(y')^c \neq \emptyset$. In this case, the transfer between the principal and the agent is M and the hypothesis (to be contradicted) becomes that:

$$u(x',\theta) - M > V(\theta). \tag{129}$$

But this contradicts Equation 119.

Thus, there is an equilibrium $\hat{W}$ under $\hat{P}$ such that agents' actions follow ϕ, yielding the same final action distribution as W. Moreover, all transfers between the agent and the principal are the same. The evidence structure is unchanged, so front-end costs are unchanged; and since the final action distribution is unchanged, back-end costs are unchanged. Because of this, the principal's expected payoffs are equal under $(\mathcal{E}, P, W)$ and $(\mathcal{E}, \hat{P}, \hat{W})$ and the statement is proven. *Q.E.D.*

For intuition, consider a general mechanism that is *not* a breach mechanism, and under which some agent takes an action z by promising y, deviating from this promise, and making the payment $P(y, E(z,y))$, where $E(z,y) \subseteq \mathcal{E}(z)$ is the equilibrium evidence that the principal shows to the arbitrator. If $P(y, E(z,y))$ is the cheapest way to take action z, then it is implicitly the price that any agent who desires to do z would pay (Equation 120). The principal could instead construct a tariff in which z is directly offered at this price (see Equation 122), and then punish breaches with the maximum punishment $\bar{M}$. This gives equally powerful incentives for any agent who previously "cheated" to take action z to now do so "honestly." Thus, the class of breach mechanisms is sufficient for the principal's design problem, as they can do no better under general mechanisms. This immediately recovers our baseline analysis when limited liability does not bind ($\bar{M} = \infty$).

A full analysis of our model under nontrivial limited liability ($\bar{M} < \infty$) is beyond the scope of this paper. But we sketch here how our results would extend in that environment. By Proposition 4, it is without loss of optimality for the principal to focus on breach mechanisms. With limited liability, the principal cannot levy a punishment exceeding $\bar{M}$. Moreover, the agent can always choose $\bar{x}$ and accept the punishment $\bar{M}$. Because all agents prefer higher actions, this is the best among all ways to *violate* one's promise. Thus, it is as if the outside option of the agent is now $\bar{u}(\theta) = \max\{u(\bar{x}, \theta) - \bar{M}, 0\}$. If participation is ensured under this outside option, then the constraint that $T(y) \leq \bar{M}$ is slack. We further note that if agent $\theta \in \Theta$ prefers $(\bar{x}, \bar{M})$ over any other element of the menu, then so too does every agent $\theta' \geq \theta$. This implies that there exists a threshold type $\bar{\theta}$ such that (i) $\bar{\theta}$ is indifferent between the proposed action and payment and $(\bar{x}, \bar{M})$ and (ii) every $\theta > \bar{\theta}$ weakly prefers $(\bar{x}, \bar{M})$ to their best option in the menu. Single-crossing implies that, if (i) holds, then (ii) must hold. The relevant participation constraints for all types $\theta < \bar{\theta}$ are twofold. First, they prefer their menu selection to not participating. Second, they prefer their menu selection to $(\bar{x}, \bar{M})$: but this is guaranteed by monotonicity, single-crossing, and the definition of $\bar{\theta}$.

However, unlike the case without limited liability, we cannot easily get rid of the first participation constraint by making it bind only for the lowest type. Instead, the designer's choice of the transfer and the final action function $\phi(\theta)$ also affects the indifference condition for type $\bar{\theta}$. This does *not* yield a type-dependent IR constraint, which would require a substantially different analysis, but rather the addition of a single constraint to capture indifference of $\bar{\theta}$. Thus, we can use a Lagrangian approach to take into account this additional constraint and maximize a modified version of the principal's virtual surplus that takes into account the corresponding Lagrange multiplier. Up to this multiplier, the problem is then still equivalent to the pointwise maximization problem we have exploited in our arguments. This in turn implies that, in the relevant space of types $\theta \in [0, \bar{\theta}]$ that do not choose to take the largest action and the penalty $\bar{M}$, the optimal final action function will have the same properties we have derived in our results.[23]

E. PROBABILISTIC ARBITRATION

In our main analysis, we assumed that the process of determining breach of contract was deterministic and error-free. However, it is natural to imagine that an arbitrator may make mistakes in judging the evidence. These mistakes may further be of two varieties: type I errors, where the agent is adjudged to have violated the contract when the true evidence does not support that conclusion; and type II errors, where the agent is adjudged to have not violated the contract when the evidence in fact proves that they did. In this extension, we show that our main analysis and results extend to a setting in which punishments are bounded and an arbitrator is subject to both such type I and type II errors.

The model and timeline are modified in two ways. First, in the arbitration phase, after the agent has acted, the arbitrator can make errors. When $\mathcal{E}(x) \subseteq \mathcal{E}(y)$, *i.e.*, there is no evidence to support that the agent deviated, there is a probability $\alpha \in [0, 1)$ that the arbitrator wrongly concludes that the agent shall be punished (type I). Instead, when $\mathcal{E}(x) \cap \mathcal{E}(y)^c \neq \emptyset$, i.e., there is evidence that proves the agent deviated, there is a probability $1 - \beta \in [0, 1]$ that the arbitrator wrongly concludes that the agent shall not be punished (type II). Second, the financial punishment that the arbitrator levies is given by some $\bar{P} < \infty$. This ensures that the mere possibility of type I errors does not dissuade all agents from participating.

An agent who acts in accordance with the contract expects to lose $\alpha \bar{P}$ in utility from being wrongfully punished, in addition to forfeiting their promised payment. An agent who fails

[23] We omit further details in the interests of space, but such details are available on request.

to act in accordance with the contract expects to lose $\beta \bar{P}$ in utility. Moreover, the best final action conditional on committing a breach is the highest-payoff action. As we show below, a sufficiently large punishment and probability of detection suffices to ensure that no agent wishes to do this, rendering type II errors irrelevant to our analysis.

The more delicate issue is the possibility of type I errors. In their presence, it is as if the agent (conditional on participation) has the payoff $u(x,\theta) - (1-\alpha)t - \alpha\bar{P}$ and the principal has the payoff $\pi(x,\theta) + (1-\alpha)t + \alpha\bar{P}$. It may now be optimal to exclude some agents from participation in the mechanism. However, conditional on this optimal choice of exclusion, our remaining analysis is unchanged.

These claims are summarized in the following Proposition, which makes use of the definition $\bar{u} = \max_{x\in X,\theta\in\Theta} u(x,\theta)$:

PROPOSITION 5: *If $(\beta - \alpha)\bar{P} > (1-\alpha)\bar{u}$, then the claims of Theorems 1 and 2,*[24] *and Corollary 1 hold.*

PROOF: We first show that no agent will participate in the mechanism and then breach the contract under the assumption that $(\beta-\alpha)\bar{P} > (1-\alpha)\bar{u}$. For an agent who selects $(y, T(y))$, compliance entails an expected payment of $\alpha\bar{P} + (1-\alpha)T(y)$, whereas a detectable breach entails an expected payment of $\beta\bar{P} + (1-\beta)T(y)$. Suppose, toward a contradiction, that an agent participates and chooses a detectable breach $x \notin C(y)$. Let $u_B = u(x,\theta) \le \bar{u}$. Participation requires $U_B := u_B - \beta\bar{P} - (1-\beta)T(y) \ge 0$. If $\beta < 1$, this implies $(1-\beta)\big(\bar{P} - T(y)\big) \ge \bar{P} - u_B$. The agent could instead comply by choosing $x' \in C(y)$, yielding $U_C := u_C - \alpha\bar{P} - (1-\alpha)T(y)$, where $u_C := u(x',\theta) \ge 0$. The difference between the payoff from compliance and that from breach satisfies

$$
\begin{aligned}
(1-\beta)(U_C - U_B) &= (1-\beta)(u_C - u_B) + (1-\beta)(\beta-\alpha)(\bar{P} - T(y)) \\
&\ge (1-\beta)u_C - (1-\alpha)u_B + (\beta-\alpha)\bar{P} \\
&\ge (\beta-\alpha)\bar{P} - (1-\alpha)\bar{u} > 0.
\end{aligned}
$$

where the second line plugs in $(1-\beta)T(y) \le u_B - \beta\bar{P}$ (from participation) and simplifies, the third line uses $u_B \le \bar{u}$ and $u_C \ge 0$, and the final inequality asserts the assumed condition. The agent therefore strictly prefers compliance, a contradiction. If $\beta = 1$, the stated condition reduces to $\bar{P} > \bar{u}$, in which case every detectable breach gives the agent a strictly negative payoff. Hence, no agent ever participates and then commits a detectable breach. Type II errors consequently do not occur on path and are immaterial to the remainder of the analysis.

We now study how type I errors affect the principal's problem. As discussed before, these transform the principal and agent gross payoffs into the following form $\tilde{\pi}(x,\theta) = \pi(x,\theta) + \alpha\bar{P}$ and $\tilde{u}(x,\theta) = u(x,\theta) - \alpha\bar{P}$ conditional on the agent participating. This violates our maintained assumption that $\tilde{\pi}(0,\theta) = \tilde{u}(0,\theta) = 0$ for all $\theta \in \Theta$ but does not affect any of our other maintained assumptions on the gross payoffs. We must therefore adapt Lemma 3 to handle this change. A mechanism now comprises (ϕ, ξ, r, T), where $r : \Theta \to \{0,1\}$ is a mapping from types to inclusion, with $r(\theta) = 1$ denoting inclusion and $r(\theta) = 0$ denoting exclusion. The agent's IR constraint now changes to $u(\phi(\theta),\theta) - (1-\alpha)T(\xi(\theta)) \ge \alpha\bar{P}$ if $r(\theta) = 1$. The IC constraint is the same, except the agent's payoff is replaced by $u(x,\theta) - (1-\alpha)T(y)$ and it only need hold when $r(\theta) = 1$. The Obedience constraint is unchanged, except that it need only

[24] Here, the definition of non-triviality must be strengthened to include the hypothesis that $H(\phi^P(\theta),\theta) > 0$ at some θ for which there is a unique, non-degenerate, and interior maximum.

hold when $r(\theta) = 1$. Finally, we require that the agent's decision not to participate is incentive compatible:

$$r(\theta) = 0 \implies \mathcal{L}(\theta) \equiv \max_{y \in X, x \in C(y)} u(x, \theta) - (1-\alpha)T(y) - \alpha \bar{P} \leq 0. \tag{130}$$

$\mathcal{L}(\theta)$ is a weakly increasing function of θ. Hence, $r(\theta)$ may be taken without loss of optimality as monotone increasing, and so there exists a $\bar{\theta} \in [0,1]$ such that $r(\theta) = \mathbb{I}[\theta \geq \bar{\theta}]$.

Lemmas 8 and 9 must be updated to replace the IR constraints with the modified IR constraint. Lemma 9 holds as stated except that the tariff must instead be given, for all $x \in X$ such that there exists $\theta \in \Theta$ with $r(\theta) = 1$ and $\phi(\theta) = x$, by:

$$(1-\alpha)T(x) = Z + u(x, \phi^{-1}(x)) - \int_{\bar{\theta}}^{\phi^{-1}(x)} u_\theta(\phi(s), s)\,\mathrm{d}s \tag{131}$$

with $Z \leq -\alpha \bar{P}$. In Lemma 10, the principal's problem then becomes:

$$\begin{aligned}
&\max_{\phi, \bar{\theta}, Z \leq -\alpha \bar{P}} \int_{\bar{\theta}}^{1} \left(\pi(\phi(\theta), \theta) + \alpha \bar{P} + Z + u(\phi(\theta), \theta) - \int_{\bar{\theta}}^{\theta} u_\theta(\phi(s), s)\,\mathrm{d}s - \int_{\phi(\theta)}^{\overline{x}} g_b(\phi(\theta), z)\,\mathrm{d}z \right) \mathrm{d}F(\theta) \\
&\text{s.t.} \quad \phi(\theta') \geq \phi(\theta),\ \phi(\theta) \in \overline{D} \quad \forall \theta, \theta' \in \Theta : \theta' \geq \theta \\
&\qquad u(\phi(\theta), \theta) - \left(\alpha \bar{P} + Z + u(\phi(\theta), \theta) - \int_{\bar{\theta}}^{\theta} u_\theta(\phi(s), s)\,\mathrm{d}s \right) \geq 0 \quad \forall \theta \geq \bar{\theta}.
\end{aligned} \tag{132}$$

It follows that it is optimal to set $Z = -\alpha \bar{P}$. Adapting the standard integration by parts argument, we then obtain that:

$$\mathcal{R}(\overline{D}) = \max_{\bar{\theta}} \int_{\bar{\theta}}^{1} \left(\max_{x \in \overline{D}} H(x, \theta) \right) \mathrm{d}F(\theta)\,. \tag{133}$$

The fact that H is strictly supermodular is what allows us to drop the monotonicity constraint. Let $\bar{\Theta}$ denote the set of types such that $\check{H}(\theta) \equiv \max_{x \in \overline{D}} H(x, \theta) = 0$. As H is supermodular, $\check{H}$ is increasing. Moreover, for all $\theta \geq \max \bar{\Theta}$, $\check{H}$ is strictly increasing. This follows because for any $\theta, \theta' \geq \max \bar{\Theta}$ such that $\theta' > \theta$, we have that $\check{H}(\theta') = H(x^*(\theta'), \theta') \geq H(x^*(\theta), \theta') > H(x^*(\theta), \theta) = \check{H}(\theta)$, where the strict inequality follows from the fact that H is strictly increasing in θ for $x > 0$. With this, we may take $\bar{\theta} = \max \bar{\Theta}$ if $\bar{\Theta}$ is non-empty. Otherwise, if $\check{H}(\theta) < 0$ for all $\theta \in \Theta$ we take $\bar{\theta} = 1$ and if $\check{H}(\theta) > 0$ for all $\theta \in \Theta$ we take $\bar{\theta} = 0$. By non-triviality (with the stated strengthening in Footnote 24), the same arguments as those of Theorem 2 imply that any $\overline{D}$ that maximizes $\mathcal{R}$ must be infinite.

We now establish that the conclusion of Lemma 7 holds as stated. We define $\bar{\theta}(\overline{D})$ as the solution $\bar{\theta}$ that we just described. We also define $\hat{x}(\overline{D}) \equiv \max\left(\arg\max_{x \in \overline{D}} H(x, \bar{\theta}(\overline{D}))\right)$. It is immediate that if $\overline{D} \cap (0, \hat{x}(\overline{D})) \neq \emptyset$, then $\overline{D} \setminus (0, \hat{x}(\overline{D}))$ yields a strictly lower front-end cost and the same value of $\mathcal{R}$. Hence, any optimal $\overline{D}$ contains no elements between 0 and $\hat{x}(\overline{D})$. Moreover, for any (a, b) such that $\hat{x}(\overline{D}) < a < b$, we have that $\bar{\theta}(\overline{D} \setminus (a, b)) = \bar{\theta}(\overline{D})$. Thus, the arguments of Lemma 7 hold as stated. Theorem 1 and Corollary 1 follow. *Q.E.D.*

REMARK 3: This proof addresses the variant of the model that we flagged in Footnote 9. In particular, if we allow the principal not to pay back-end costs for the agents that it excludes, then it may wish to exclude agents whenever $\max_{x \in X} H(x, \theta) < 0$. The analysis above shows that this does not affect how front and back-end costs affect optimal contracts.

F. EXTENDING THE RESULTS TO MORE GENERAL COSTS

In this appendix, we describe a richer family of front-end and back-end cost functions and provide general conditions under which the conclusions of Theorems 1 and 2 hold.

F.1. *Generalizing Front-End Costs*

We first define an order over regular evidentiary correspondences, the set of which we denote by $\mathscr{E}$. We say that $\mathcal{E}'$ generates more refined evidence than $\mathcal{E}$, which we denote by $\mathcal{E}' \succsim \mathcal{E}$, if for all $x, y \in X$ such that $\mathcal{E}'(x) \subseteq \mathcal{E}'(y)$ we also have that $\mathcal{E}(x) \subseteq \mathcal{E}(y)$. In words, this means that $\mathcal{E}'$ generates more refined evidence than $\mathcal{E}$ if every time that x cannot be proven by the principal to be inconsistent with y using evidence generated by $\mathcal{E}'$, the same is true if evidence were generated by $\mathcal{E}$. Observe that $\mathcal{E}' \succsim \mathcal{E}$ if and only if $C_{\mathcal{E}'} \subseteq C_{\mathcal{E}}$. Thus, this order over evidence is equivalent to inducing a greater set of self-enforcing recommendations.

We say that a (front-end) cost function $\Gamma : \mathscr{E} \to \mathbb{R}_+$ is monotone if whenever $\mathcal{E}' \succsim \mathcal{E}$, then we have that $\Gamma(\mathcal{E}') \geq \Gamma(\mathcal{E})$. We argue that this is a natural property for a cost to possess: if an evidentiary correspondence generates more refined evidence, then it incurs a higher cost. We now show that monotonicity of Γ justifies writing costs directly over contractibility correspondences. Recall that we let $\mathcal{C}$ denote the set of regular contractibility correspondences. Formally, we define what it means for a cost function to be measurable in the induced contractibility correspondence as:

DEFINITION 5—C-measurability: *Γ is $C-$measurable if there exists a $\check{\Gamma} : \mathcal{C} \to [0, \infty]$ such that for all $\mathcal{E} \in \mathscr{E}$, we have that $\Gamma(\mathcal{E}) = \check{\Gamma}(C_{\mathcal{E}})$.*

If Γ is $C-$measurable, we call the corresponding $\check{\Gamma}$ the induced cost. We have that:

LEMMA 15: *If Γ is monotone, then it is $C-$measurable.*

PROOF: Fix an arbitrary pair of evidentiary correspondences $\mathcal{E}, \mathcal{E}' \in \mathscr{E}$ such that $C_{\mathcal{E}} = C_{\mathcal{E}'}$. We have that $C_{\mathcal{E}} \subseteq C_{\mathcal{E}'}$, which implies that $\mathcal{E} \succsim \mathcal{E}'$. By monotonicity of Γ we have that $\Gamma(\mathcal{E}) \geq \Gamma(\mathcal{E}')$. By the reverse argument, we have that $\Gamma(\mathcal{E}') \geq \Gamma(\mathcal{E})$. Hence, we have that $\Gamma(\mathcal{E}') = \Gamma(\mathcal{E})$ for all $\mathcal{E}, \mathcal{E}' \in \mathscr{E}$ that induce the same contractibility correspondence. Take $\check{\Gamma}(C_{\mathcal{E}}) = \Gamma(\mathcal{E})$. Thus, Γ is $C-$measurable with induced $\check{\Gamma}$. *Q.E.D.*

This result implies that whenever costs are monotone in the natural sense over underlying evidentiary correspondences, it is without loss of optimality to write the cost in the space of contractibility correspondences directly. It is immediate to see that all the front-end costs of distinguishing that we considered in the main text are monotone.

As in the main analysis, we call a front-end evidentiary cost of distinguishing linear if $g_f(x, y) = \kappa > 0$ and we write this cost function as Γ^κ. We can now define strong monotonicity:

DEFINITION 6—Strong Monotonicity: *A cost function Γ is strongly monotone if there exists $\varepsilon > 0$ such that, for all $\mathcal{E}, \mathcal{E}' \in \mathscr{E}$ such that $\mathcal{E}' \succsim \mathcal{E}$:*

$$\Gamma(\mathcal{E}') - \Gamma(\mathcal{E}) \geq \varepsilon \left(\Gamma^1(\mathcal{E}') - \Gamma^1(\mathcal{E})\right). \tag{134}$$

With this, we show that strong monotonicity implies a certain property for the induced cost defined over contractibility correspondences:

PROPOSITION 6: *If Γ is strongly monotone with constant $\varepsilon > 0$, then the induced $\check{\Gamma}$ satisfies:*

$$\check{\Gamma}(C') - \check{\Gamma}(C) \geq \varepsilon \left(\check{\Gamma}^1(C') - \check{\Gamma}^1(C) \right), \quad \forall C' \subseteq C \tag{135}$$

PROOF: We first observe that strong monotonicity of Γ implies monotonicity of Γ. Thus, by Lemma 15, we have that Γ is $C-$measurable and therefore has an induced $\check{\Gamma}$. Now fix an arbitrary pair $\mathcal{E}, \mathcal{E}' \in \mathscr{E}$ such that $\mathcal{E}' \succsim \mathcal{E}$, we have that:

$$\check{\Gamma}(C_{\mathcal{E}'}) - \check{\Gamma}(C_{\mathcal{E}}) = \Gamma(\mathcal{E}') - \Gamma(\mathcal{E}) \geq \varepsilon \left(\Gamma^1(\mathcal{E}') - \Gamma^1(\mathcal{E}) \right) = \varepsilon \left(\check{\Gamma}^1(C_{\mathcal{E}'}) - \check{\Gamma}^1(C_{\mathcal{E}}) \right), \tag{136}$$

where the first equality is by $C-$measurability, the first inequality is by strong monotonicity of Γ and the final equality is by definition of the linear front-end cost of distinguishing. Thus, as $\mathcal{E}' \succsim \mathcal{E} \iff C_{\mathcal{E}'} \subseteq C_{\mathcal{E}}$, we have established the required condition on $\check{\Gamma}$. *Q.E.D.*

By Lemma 12, we have that $C \in \mathcal{C}$ if and only if $C(y) = [\underline{\delta}_C(y), \overline{\delta}_C(y)]$ for two semicontinuous, monotone functions with the escalator-step property. We say that $\check{\Gamma} : \mathcal{C} \to \mathbb{R}_+$ is continuous if it is continuous in the product L_1-topology with respect to $(\underline{\delta}_C, \overline{\delta}_C)$.

Given a monotone front-end cost function Γ, we say that Γ is continuous if its induced $\check{\Gamma}$ is continuous. In this case, Proposition 3 implies that the induced cost function over the set of self-enforcing recommendations is continuous in the Hausdorff topology. We now show that strong monotonicity and continuity of Γ are sufficient for the conclusion that optimal contracts are necessarily coarse.

PROPOSITION 7: *Suppose that Γ is continuous and strongly monotone with constant $\varepsilon > 0$ and the principal faces no back-end costs. An optimal evidence structure exists and every optimal evidence structure is coarse. Moreover, the bound on the richness of any optimal evidence structure and menu from Theorem 1 holds, with ε replacing $\underline{g}_f$.*

PROOF: By continuity, Lemma 4 applies and so an optimal evidence structure exists. By Proposition 6, Lemma 6 holds with ε replacing $\underline{g}_f$. The remaining arguments follow the proof of Theorem 1. *Q.E.D.*

F.2. *Generalizing Back-End Costs*

We consider back-end cost functions $\Gamma : \mathscr{E} \times \Delta(X) \to \mathbb{R}_+$ given by:

$$\Gamma(\mathcal{E}, Q) = \int_X \gamma(\mathcal{E}, x) \, \mathrm{d}Q\,(x), \tag{137}$$

for some $\gamma : \mathscr{E} \times X \to \mathbb{R}_+$. The interpretation here is that $\gamma(\mathcal{E}, x)$ is the cost of generating the evidence according to the action x given the evidence structure $\mathcal{E}$. Given this, Γ is then the expected cost that will be incurred if the distribution over actions that is taken is given by Q. We say that γ is monotone if whenever $\mathcal{E}' \succsim \mathcal{E}$, then we have that $\gamma(\mathcal{E}', x) \geq \gamma(\mathcal{E}, x)$ for all $x \in X$. By the exact same arguments as those in Lemma 15, if γ is monotone, then it is C-measurable. We denote by $\tilde{\gamma}$ the induced cost over contractibility correspondences, *i.e.,* $\tilde{\gamma}(C_{\mathcal{E}}, x) \equiv \gamma(\mathcal{E}, x)$. By monotonicity of γ in $\mathcal{E}$, we also have that $\tilde{\gamma}$ is monotone with respect to set inclusion in the space of regular contractibility correspondences, that is, $C' \subseteq C$ implies that $\tilde{\gamma}(C', x) \geq \tilde{\gamma}(C, x)$ for all $x \in X$. With this, we have that the induced back-end cost is also C-measurable and we denote the induced cost over contractibility correspondences by $\tilde{\Gamma}$, *i.e.,* $\tilde{\Gamma}(C_{\mathcal{E}}, Q) =$

$\int_X \tilde{\gamma}(C_{\mathcal{E}}, x)\,\mathrm{d}Q\,(x) = \Gamma(\mathcal{E}, Q)$. We say that $\tilde{\gamma}$ is *separable* if $\tilde{\gamma}(C,x) = \hat{\gamma}(C(x), x)$ for some $\hat{\gamma} : \Xi \times X \to \mathbb{R}_+$, where $\Xi \subseteq 2^X$ is the collection of closed and convex subsets of X, and we endow it with the Hausdorff topology. We note that as $\tilde{\gamma}$ is monotone with respect to set inclusion, so too is $\hat{\gamma}$. We say that a separable $\tilde{\gamma}$ is continuous if the induced function $\hat{\gamma}$ is continuous with respect to the product topology. Finally, we define $\zeta(x) = \hat{\gamma}([0,x], x)$ and $H(x,\theta) = J(x,\theta) - \zeta(x)$. With these definitions, we can state the following result.

PROPOSITION 8: *Suppose that the principal faces no front-end costs and γ is monotone with an induced $\tilde{\gamma}$ that is separable and continuous. An optimal evidence structure $\mathcal{E}^*$ with cardinality $|\mathcal{E}^*(X)| = \infty$ exists. Moreover, if H is non-trivial and twice continuously differentiable, then every optimal evidence structure $\mathcal{E}^*$ and every optimal menu $\mathcal{M}^*$ have cardinality $|\mathcal{E}^*(X)| = |\mathcal{M}^*| = \infty$.*

PROOF: If γ is monotone with a separable $\tilde{\gamma}$, then we may write the back-end cost as:

$$\tilde{\Gamma}(C,Q) = \int_X \hat{\gamma}(C(x), x)\,\mathrm{d}Q\,(x) = \int_X \hat{\gamma}([\underline{\delta}(x), \overline{\delta}(x)], x)\,\mathrm{d}Q\,(x), \tag{138}$$

where $\underline{\delta}$ and $\overline{\delta}$ are the lower and upper envelopes of C from the representation of Lemma 12. The arguments of Lemma 2 then apply here and we have that $\underline{\delta}(x) \equiv 0$ is optimal. Thus:

$$\begin{aligned}\tilde{\Gamma}(C,Q_\phi) &= \int_X \hat{\gamma}([0,\overline{\delta}(x)], x)\,\mathrm{d}Q_\phi(x) = \int_\Theta \hat{\gamma}([0,\overline{\delta}(\phi(\theta))], \phi(\theta))\,\mathrm{d}F(\theta) \\ &= \int_\Theta \hat{\gamma}([0,\phi(\theta)], \phi(\theta))\,\mathrm{d}F(\theta) = \int_\Theta \zeta(\phi(\theta))\,\mathrm{d}F(\theta)\,,\end{aligned} \tag{139}$$

where the penultimate line follows (as in Lemma 3) from Obedience and the last uses the definition of $\zeta(x) = \hat{\gamma}([0,x], x)$. We can now apply the remaining arguments of Lemma 3 where the statement now holds with $H(x,\theta) = J(x,\theta) - \zeta(x)$. By the hypothesis of continuity of $\tilde{\gamma}$, we have that ζ (and therefore H) is continuous. From this, the arguments from Appendix A.2 apply and the conclusion of Theorem 2 then follows. For the second conclusion, non-triviality and twice continuous differentiability yield the conclusion by the arguments of Theorem 2. *Q.E.D.*

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