

The Robust Properties of Economies with Endogenous Production Networks

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Abstract

We characterize equilibria in a canonical class of economies with endogenous production networks, showing that log-prices equal a min-max over networks of a primitive affine function of log-productivities. This formula captures how resilience and fragility interact: the “min” reflects resilience from substituting suppliers, while the “max” reflects fragility from complementarity-induced rigidity. Applying these results, we generalize Hulten’s theorem, finding that local Domar weights are generally *insufficient* to describe propagation of even small shocks. Moreover, we characterize how microeconomic volatility affects GDP, finding that smooth changes in Domar weights overstate its adverse effects by omitting option value from switching suppliers.

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1 Introduction

Modern production is characterized by the presence of complex supply chains. What determines their structure? How does the production network react to economic shocks? What are the resulting aggregate implications of changes in microeconomic productivity and its volatility? Currently, there are two broad perspectives on the economic implications of networked production. On the one hand, there is the view that rigid supply chains involved in networked production make the economy *fragile* (see *e.g.*, Elliott, Golub, and Leduc, 2022; Acemoglu and Tahbaz-Salehi, 2025). On the other hand, there is a competing view that the ability to change suppliers (see *e.g.*, Dew-Becker, 2023; Kopytov, Mishra, Nimark, and Taschereau-Dumouchel, 2024) may make the economy *resilient* or anti-fragile (Taleb, 2013).

Answering these questions requires modeling the determinants of the production network, going beyond what can be captured by classic production network models with Cobb-Douglas technology (Long Jr and Plosser, 1983; Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Salehi, 2012; Carvalho and Tahbaz-Salehi, 2019) or log-linearizations of more general models (Hulten, 1978; Liu, 2019). However, this presents a well-recognized theoretical challenge: the fixed point system characterizing equilibrium ceases to be (log-)linear. As a result, there is no known general solution for equilibrium prices, output, and the production network.

To overcome this challenge, the literature has proceeded in three directions, each yielding valuable insights: (i) specializing to specific production technologies or studying specific properties of equilibrium (Oberfield, 2018; Acemoglu and Azar, 2020; Elliott et al., 2022; Kopytov et al., 2024), (ii) considering higher-order approximations (Baqee and Farhi, 2019; Carvalho, Nirei, Saito, and Tahbaz-Salehi, 2021), or (iii) studying the asymptotic effects of shocks (Dew-Becker, 2023). Thus, while our understanding of endogenous production networks has advanced enormously, we lack a general theory that is valid outside of important but special cases (*cf.* (i)) or for shocks that are neither very small (*cf.* (ii)) nor very large (*cf.* (iii)).

In this paper, we leverage tools from decision theory (see *e.g.*, Maccheroni, Marinacci, and Rustichini, 2006; Cerreia-Vioglio, Corrao, and Lanzani, 2026) to study the robust properties of the canonical static single-factor production network model (as in Acemoglu et al., 2012; Carvalho and Tahbaz-Salehi, 2019). That is, we characterize the global properties of equilibrium prices and the production network structure, while allowing for rich technological possibilities and endogenous supply relationships at the microeconomic level. We maintain the core assumption at the heart of input-output analysis (Leontief, 1941): production is constant returns to scale (CRS). We otherwise make no parametric assumptions on technology.

Our main result provides a complete characterization of equilibrium prices and output in this family of production network models. As part of this characterization, we provide globally valid and primitive formulae for equilibrium prices and output that hold for any such economy.

We exploit these formulae to understand conditions on technology at the microeconomic level such that resilience and/or fragility (by various definitions) emerge as endogenous macroeconomic properties. In so doing, we also derive a number of new results in producer theory that may be of independent interest. We further apply the results to provide a generalization of [Hulten’s \(1978\)](#) theorem compatible with non-differentiable production technologies—which we show arise naturally from endogeneity of the production network. Importantly, this result demonstrates that local Domar weights can be uninformative for the propagation of even small shocks. Thus, in contrast to the standard interpretation of [Hulten \(1978\)](#), the technology and network structure of the economy can matter, even locally. Moreover, we characterize the effects of microeconomic volatility on average output, finding generally non-monotone effects and that local changes in Domar weights necessarily *overstate* the adverse consequences of increased risk as they omit a form of option value that results from switching suppliers.

Model. The model follows the canonical static single-factor production network setting (see [Carvalho and Tahbaz-Salehi, 2019](#), for a review). There are many sectors, each perfectly competitive, that use as inputs the outputs of other sectors and a single primary factor, which is inelastically supplied. We allow for an arbitrary CRS production function. Thus, our analysis nests: (i) the Cobb-Douglas benchmark ([Acemoglu et al., 2012](#)), (ii) nested constant elasticity of substitution (CES) technology ([Baqaee and Farhi, 2019](#); [Dew-Becker, 2023](#)), (iii) heterogeneous CES technology ([Hanoch, 1971](#); [Chodorow-Reich, Gabaix, and Viviano, 2026](#)), (iv) Cobb-Douglas production technology with endogenous supplier choice at the extensive and intensive margins ([Acemoglu and Azar, 2020](#); [Kopytov et al., 2024](#)), (v) Leontief production technology with and without endogenous supplier choice ([Acemoglu and Tahbaz-Salehi, 2025](#)), and (vi) task-based production functions ([Acemoglu and Restrepo, 2018](#)).

Main Characterization Results. We characterize all mappings from productivity to prices that are possible in production network economies in terms of simple functional properties. This “anything goes” result shows that equilibrium prices need, absent knowledge of the economy’s microeconomic technology, satisfy only weak properties. However, for any given technology, our result provides a globally valid formula describing equilibrium: the equilibrium log-price vector is the “min-max” over pairs of input-output networks of the Leontief inverse of the maximizing network applied to the vector of log-productivities plus an explicit wedge that depends on technology. That is, equilibrium prices are always the “min-max” of what prices would be in Cobb-Douglas economies with a certain productivity distortion. This result forms the basis of our subsequent applications, in which we: (i) unpack the economic interpretation of the “min-max” structure in terms of the competing forces of resilience and fragility, (ii) characterize the propagation of microeconomic productivity shocks into aggregate prices and output, and (iii) characterize the effects of changing microeconomic volatility on economic activity.

To derive the main characterization, we first express the fixed-point equation for equilibrium prices that holds in such economies in terms of log-prices. In this system of equations, each sector’s log-price is linear in its log-productivity and the log of its unit cost function. Next, we show that any unit cost function that arises from a CRS production function also results from a seemingly more specific endogenous Leontief production function. Using this observation, we show that the log-cost function is a “min-max” over pairs of vectors of input weights of an *affine* function of log-prices. Finally, we show that the operations of finding the fixed point and computing the “min-max” commute. This allows us to obtain the primitive formula because, if we ignore the “min-max,” then the fixed point system is linear and can be solved analytically.

Resilience *vs.* Fragility. We then apply this general representation to understand when the economy is resilient (fragile): defined as the property that it benefits (suffers) from more dispersed microeconomic productivity shocks. We also relate the results to alternative notions of fragility and resilience, including: the ability to abate extreme shocks, large shocks being proportionately worse than small shocks, and technology being (in)elastic. The general representation makes clear that neither property should generally be expected globally: the “min” will serve to induce concavity (resilience) and the “max” will lead to convexity (fragility). Thus, prices will often be “S-shaped,” *i.e.*, concave in some regions and convex in others.

This notwithstanding, many standard specifications of production functions ensure that the economy is globally resilient or fragile. We show that this occurs when each sector’s log-cost function is a concave or convex function of log-prices, respectively. As a concrete example, a CES production function with elasticity of substitution larger (smaller) than one has a globally concave (convex) log-cost function. In such cases, the specification of technology necessarily rules out *qualitative* phenomena: increases in productivity volatility are always either good or bad on average. For this reason, second-order approximations rule out “S-shaped” responses.

To understand when assumptions on technology can rule out these qualitative phenomena, we characterize the property that a log-cost function is globally concave or convex in terms of corresponding production functions. We show that any concave log-cost function arises from an endogenous Cobb-Douglas production function (as employed by, *e.g.*, [Acemoglu and Azar, 2020](#); [Kopytov et al., 2024](#)). This shows that endogenous Cobb-Douglas technology is both more general and more restrictive than it may appear: it captures *any* resilient technology, but only resilient technology. Moreover, we show that any convex log-cost function arises from a particular form of task-based production (see *e.g.*, [Acemoglu and Restrepo, 2018](#)) where a set of tasks, each of which has Cobb-Douglas technology, must be completed.

A Generalized [Hulten’s \(1978\)](#) Theorem. We next apply our representation to derive the effects of small shocks to productivity on equilibrium prices and output. We show that equilibrium prices are always directionally differentiable, but not necessarily Gateaux differ-

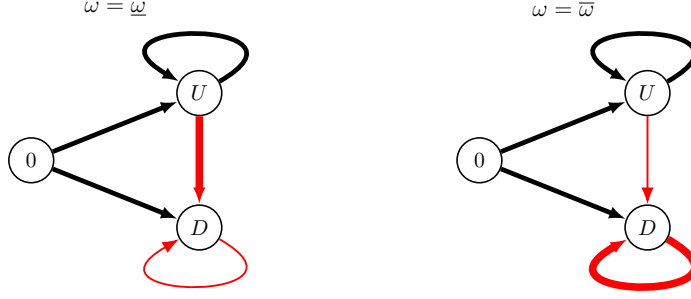
entiable, *i.e.*, the derivative depends on the current productivity as well as on the direction of shock considered. The failure of Gateaux differentiability is economically important because it corresponds to the presence of technology switching: if a sector is ever indifferent at equilibrium prices between producing using different input mixes, then different directions of shocks will lead to different input mixes and different shock propagation. While such indifferences are non-generic (*i.e.*, they hold on a zero measure set of productivities), ignoring the non-differentiabilities they induce can lead to inaccurate quantitative guidance, as we show in simple examples. Intuitively, when such a switch happens, the correct coefficients in any order of Taylor expansion jump and the previously accurate approximation suddenly fails.

We use this formula for the derivative of prices to obtain a corresponding formula for the derivative of GDP, providing a generalization of Hulten’s theorem: the effect on GDP is the *maximum* over the set of Domar weights (the vector of sales shares of nominal GDP across sectors) that may prevail at equilibrium of the inner product of these Domar weights and the log-productivity shocks. The presence of the maximum generalizes Hulten’s theorem beyond the differentiable case and allows economically for the presence of technology switching. The result implies that observed Domar weights do not serve as sufficient statistics for describing the first-order effects of productivity shocks on output in general. This provides nuance to the typical interpretation of Hulten’s theorem as implying that, for small shocks, input-output structure can be ignored. Instead, one must know which alternative technologies are available.

The Macroeconomic Effects of Microeconomic Volatility. Finally, we apply the results to study the effects of changes in volatility of microeconomic productivity on average GDP, which we call the local resilience of the economy. We provide a general formula for local resilience as the sum of two terms: the average derivative of the Domar weights in productivity—which may be positive or negative—and a term reflecting the average jump in Domar weights from technology switching—which is always positive. The first term reflects an *exact* version of the [Baqae and Farhi \(2019\)](#) logic: derivatives of Domar weights describe the second-order effects of productivity shocks. The second term reveals an additional force: when technology switches can occur, there is a form of option value in the economy, which always pushes toward resilience. Moreover, the presence of this term underscores the importance of technology switching; even if these switches are rare, that does not mean that they can be neglected.

Outline. The rest of the paper proceeds as follows. Section 2 provides a simple example to illustrate challenges in understanding the global properties of models with endogenous production networks. Section 3 introduces the model. Section 4 provides the main characterization results. Section 5 studies when economic outcomes feature resilience or fragility. Section 6 characterizes the local effects of shocks, generalizing [Hulten’s \(1978\)](#) Theorem. Section 7 characterizes the effects of changes in microeconomic volatility on GDP. Section 8 concludes.

Figure 1: Illustrating the Potential Network Configurations



Notes: The left network corresponds to D using U intensively ($\omega = \underline{\omega}$). The right network corresponds to D using D intensively ($\omega = \bar{\omega}$). The thickness of arrows denotes the strength of the connection and the direction denotes input usage.

2 Endogenous Networks: A Motivating Example

We begin with a simple example to motivate the importance of a general approach to the study of endogenous production networks. Suppose that there are two sectors in the economy, an upstream sector U and a downstream sector D . The upstream sector uses labor and upstream inputs to produce output:

$$Y_U = 2Z_U Q_{U,0}^{\frac{1}{2}} Q_{U,U}^{\frac{1}{2}} \quad (1)$$

where Z_U is the productivity of sector U , $Q_{U,0}$ is the input of labor, and $Q_{U,U}$ is sector U 's usage of goods produced in sector U . The downstream sector uses labor, upstream inputs, and downstream inputs to produce output and can do so using one of two different production technologies indexed by $\omega \in \{\underline{\omega}, \bar{\omega}\} \subset (0, 1)$:

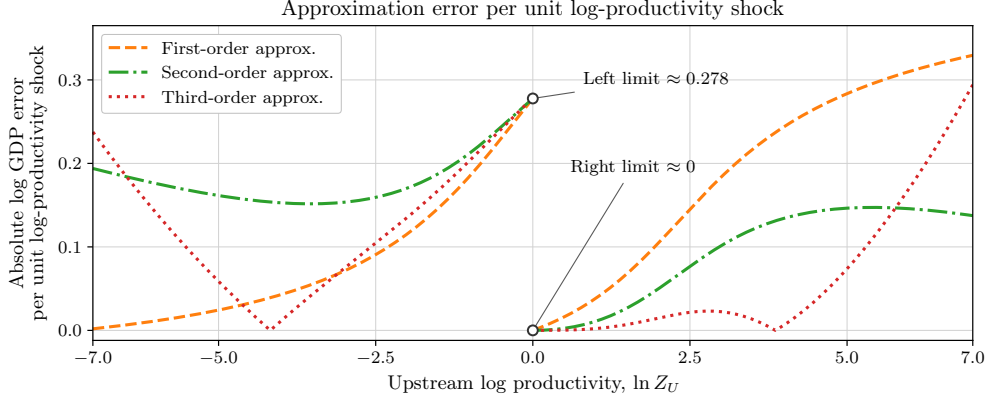
$$Y_D^\omega = 2Z_D Q_{D,0}^{\frac{1}{2}} \left(\min \left\{ \frac{Q_{D,D}}{\omega}, \frac{Q_{D,U}}{1-\omega} \right\} \right)^{\frac{1}{2}} \quad (2)$$

where Z_D is the productivity of sector D , $Q_{D,0}$ is the input of labor, $Q_{D,D}$ is D 's usage of goods produced downstream, and $Q_{D,U}$ is D 's usage of upstream inputs. Intuitively, upstream and downstream goods are always perfect complements in the production of downstream goods, but the downstream sector can choose between two distinct production technologies. We illustrate the potential network configurations that can arise in Figure 1.

To close the model, we assume that labor has an aggregate supply of one and has a price that we normalize to one. Finally, a representative household has preferences:

$$U = 2C_D^{\frac{1}{2}} C_U^{\frac{1}{2}} \quad (3)$$

Figure 2: Technology Switching Can Lead to the Failure of Local Methods, Even Locally



Notes: This figure plots the absolute error from approximating $\ln \text{GDP}$ per unit of upstream log-productivity, for first- (orange), second- (green), and third-order (red) Taylor approximations to $\ln \text{GDP}$ in $\ln Z_U$ around the point $\ln Z_U = 0$ from the right. Parameters: $\underline{\omega} = 0.2$, $\bar{\omega} = 0.8$, $\ln Z_D = 0$.

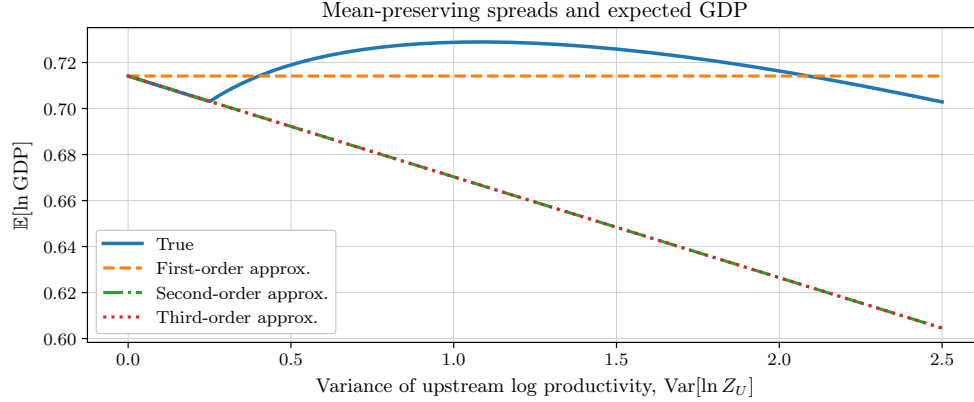
where C_i is final consumption from sector $i \in \{D, U\}$. It is simple to show that the consumer's log-utility in competitive equilibrium (which we will call $\ln \text{GDP}$) is:

$$\ln \text{GDP} = \frac{1}{2} \ln 2 + \ln Z_U + \ln Z_D - \frac{1}{2} \begin{cases} \ln \left(\underline{\omega} + \sqrt{\underline{\omega}^2 + 4(1 - \underline{\omega}) \frac{Z_D^2}{Z_U^2}} \right), & Z_D \leq Z_U, \\ \ln \left(\bar{\omega} + \sqrt{\bar{\omega}^2 + 4(1 - \bar{\omega}) \frac{Z_D^2}{Z_U^2}} \right), & Z_D > Z_U. \end{cases} \quad (4)$$

We now make two observations that will motivate our general approach. First, $\ln \text{GDP}$ is not a differentiable function of productivity. In particular, because the downstream sector's technology—and hence the production network—switches at points $Z_D = Z_U$, standard approaches based on first-, second-, or higher-order Taylor expansions can give inaccurate answers, even *locally*. We illustrate this phenomenon in Figure 2, where the proportional approximation error is discontinuous under such approximations: it is zero for small positive shocks but circa 30% for small negative ones. For this reason, our general theory directly addresses these technological switches in economies with endogenous production networks.

Second, local approaches can obscure the presence of qualitatively interesting and potentially important economic phenomena that can arise with endogenous production networks. One of the most important motivations in the modern production network literature is understanding how microeconomic volatility in any given sector may affect average GDP. By definition, first-order methods imply that microeconomic volatility has no effect on output, while to second-order the effect must be unambiguously positive or negative. To illustrate what this can obscure, Figure 3 plots the effects of increasing microeconomic volatility on average output. The global solution reveals an S-shaped response in which initial increases in variance lower expected

Figure 3: Endogenous Resilience and Fragility to Volatility



Notes: This figure plots expected $\ln$ GDP when $\ln Z_U = k \pm x$ (each with probability 0.5) against $\text{Var}[\ln Z_U] = x^2$ under the global solution (blue) and under first-, second-, and third-order Taylor approximations around the point $\ln Z_U = k$. Parameters: $\underline{\omega} = 0.2$, $\bar{\omega} = 0.8$, $\ln Z_D = 0$, $k = 0.5$.

output, intermediate levels of variance increase aggregate output, and high levels of variance decrease output again. Intuitively, at low levels of variance, the economy mostly operates one Leontief technology downstream, and so variance in input prices is harmful. However, at intermediate levels of variance, the downstream sector can switch across technologies and the option value of switching technologies makes variance beneficial. Ultimately though, when variance becomes very large, the fact that they always operate a Leontief technology (even if they can choose the best one) makes variance lower average output. In this way, the ultimate resilience or fragility of the macroeconomy to changes in volatility is an endogenous property.

3 Model

We consider a canonical production network model (see [Acemoglu et al., 2012](#); [Carvalho and Tahbaz-Salehi, 2019](#); [Baqae and Farhi, 2019](#); [Acemoglu and Azar, 2020](#); [Dew-Becker, 2023](#)).

3.1 Primitives

Sectors. Each sector $i \in [n] = \{1, \dots, n\}$ produces output according to a production function $F_i : \mathbb{R}_+^{n+1} \rightarrow \mathbb{R}_+$ that combines a primary input Q_{i0} and intermediate inputs $Q_i = (Q_{ij})_{j=1}^n$:

$$F_i(Q_{i0}, Q_i) = Z_i \tilde{G}_i(Q_{i0}, Q_i) \quad (5)$$

where $Z_i \in \mathbb{R}_{++}$ is productivity and $\tilde{G}_i : \mathbb{R}_+^{n+1} \rightarrow \mathbb{R}_+$ is the underlying production technology.

The goal of our analysis is to study the *robust* properties of equilibrium outcomes in production networks, *i.e.*, those properties that obtain without functional form restrictions on the nature of technology. Thus, we impose minimal regularity assumptions and the core economic assumption from input-output analysis (Leontief, 1941): constant returns-to-scale (CRS).

Assumption 1. *For all sectors $i \in [n]$, the production function $\tilde{G}_i : \mathbb{R}_+^{n+1} \rightarrow \mathbb{R}_+$ is positive homogeneous, non-decreasing, continuous, and such that the optimal expenditure share on the primary factor is bounded away from zero.¹*

In order to simplify exposition for the main text, we express production technologies as:

$$\tilde{G}_i(Q_{i0}, Q_i) = \left(\frac{Q_{i0}}{1 - \beta} \right)^{1-\beta} \left(\frac{G_i(Q_{i1}, \dots, Q_{in})}{\beta} \right)^\beta \quad (6)$$

where $G_i : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ satisfies all but the last of the requirements of Assumption 1 and $\beta \in (0, 1)$. We show in Lemma 14 in Appendix E that *any* economy with production technologies satisfying Assumption 1 can be *exactly* represented by technologies of this seemingly more restrictive form. Moreover, so that Z_i has an unambiguous meaning, we normalize G_i such that $\max_{Q_i \in \Delta} G_i(Q_i) = 1$, where $\Delta = \left\{ w \in \mathbb{R}_+^n : \sum_{j=1}^n w_j = 1 \right\}$. We call such a G_i *regular*.

Thus, the key object that determines the network of input-output relationships is the intermediate input production function G_i . The current literature has considered various assumptions on G_i , as we shortly detail through a number of examples. In particular, our formulation allows us to nest models in which the technologies, suppliers, and substitution elasticities of sectors are chosen endogenously.

Households. As the focus of our analysis is on the production network, we follow Acemoglu et al. (2012), Dew-Becker (2023), and Kopytov et al. (2024) in specifying household preferences and labor supply. This is justified because, as is well known and we will shortly see, demand has no role in determining equilibrium prices. Instead, demand matters only for computing GDP given prices. A representative household inelastically supplies a unit of primary factor and consumes a bundle of goods $C \in \mathbb{R}_+^n$ with a Cobb-Douglas utility function:

$$U(C) = \prod_{i=1}^n \left(\frac{C_i}{\xi_i} \right)^{\xi_i} \quad \forall C \in \mathbb{R}_+^n \quad (7)$$

where each $\xi_i \geq 0$ captures the importance of good i for the final consumer and $\sum_{i=1}^n \xi_i = 1$. Normalizing the wage in the economy to one and denoting the price of good i as P_i , the

¹Continuity is equivalent to assuming that each G_i is only upper semicontinuous since we restrict to intermediate production functions that are positive homogeneous (see, e.g., Moore, 1999, Proposition 5.95). Second, our definition of the property that the expenditure share on the primary factor is bounded away from zero is that the production function of sector i can be written for some $\bar{\beta} \in (0, 1)$ as $\tilde{G}_i(Q_0, Q) = \max_{\ell \in [0, Q_0]} ((Q_0 - \ell)/(1 - \bar{\beta}))^{1-\bar{\beta}} (\hat{G}_i(\ell, Q)/\bar{\beta})^{\bar{\beta}}$, where $\hat{G}_i$ satisfies the other criteria of Assumption 1.

household faces the standard budget constraint $\sum_{i=1}^n C_i P_i \leq 1$. In general, so long as U is positive homogeneous, our results characterizing GDP extend.²

3.2 Equilibrium

A competitive equilibrium is defined by requiring that sectors minimize costs and make zero profits, all markets clear, and households maximize utility.³

Definition 1. A competitive equilibrium is a vector of prices, $P^* \in \mathbb{R}_{++}^n$, a matrix of unitary input quantities $Q^* \in \mathbb{R}_+^{n \times (n+1)}$, a vector of production levels $Y^* \in \mathbb{R}_+^n$ and a final consumption vector $C^* \in \mathbb{R}_+^n$ such that:

$$(Q_{i0}^*, Q_i^*) \in \arg \min_{(Q_{i0}, Q_i) \in \mathbb{R}_+^{n+1}} \sum_{j=1}^n Q_{ij} P_j^* + Q_{i0} \quad s.t. \quad F_i(Q_{i0}, Q_i) \geq 1 \quad \forall i \in [n]. \quad (8)$$

$$P_i^* = \sum_{j=1}^n Q_{ij}^* P_j^* + Q_{i0}^* \quad \forall i \in [n]. \quad (9)$$

$$\sum_{i=1}^n Y_i^* Q_{i0}^* = 1, \quad \sum_{i=1}^n Y_i^* Q_{ij}^* + C_j^* = Y_j^* \quad \forall j \in [n]. \quad (10)$$

$$C^* \in \arg \max_{C \in \mathbb{R}_+^n} U(C) \quad s.t. \quad \sum_{i=1}^n C_i P_i^* \leq 1. \quad (11)$$

We call a solution to Equation 9 an equilibrium price vector.

3.3 Preliminaries: The Log-Cost Operator and Equilibrium Prices

We begin by reducing the determination of equilibrium prices to the solution of a non-linear fixed point equation. To do this, we define the cost function of sector $i \in [n]$ as:

$$K_i(P) = \min_{Q_i \in \mathbb{R}_+^n} \sum_{j=1}^n Q_{ij} P_j \quad s.t. \quad G_i(Q_i) \geq 1. \quad (12)$$

We now transform this cost function to yield the log-cost of the sector as a function of log-prices:

$$T_i(p) = \ln(K_i(\exp(p))) \quad \forall p \in \mathbb{R}^n. \quad (13)$$

We define the cost operator $K : \mathbb{R}_{++}^n \rightarrow \mathbb{R}_{++}^n$ and the log-cost operator $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ as the operators whose i -th components are, respectively, K_i and T_i . By defining the inverse

²This is because we can treat U as a production function and apply Proposition 1 below (see Footnote 5).

³These requirements are equivalent to profit maximization in the more standard definition of competitive equilibrium, with the standard caveat that a sector's price may be indeterminate if it does not produce.

log-productivity of sector $i \in [n]$ as $x_i = -\frac{1}{1-\beta} \ln Z_i$, the following Lemma characterizes the equilibrium log-price vectors, that is, $p = \ln(P)$ where P is part of a competitive equilibrium.

Lemma 1. *A vector $p \in \mathbb{R}^n$ is an equilibrium log-price vector at $x \in \mathbb{R}^n$ if and only if:*

$$p_i = (1 - \beta)x_i + \beta T_i(p) \quad \forall i \in [n]. \quad (14)$$

Moreover, there exists a unique solution to this fixed-point equation that we denote by $p^* = \Pi(x)$.

The first part follows immediately from the fact that P is an equilibrium price vector if and only if $P_i = Z_i^{-1} K_i(P)^\beta$ for all sectors $i \in [n]$ (see Lemma 8) and taking logarithms. Uniqueness follows from the fact that the operator defining the log-price fixed point equation is a contraction.⁴ With this, $\Pi(x)$ represents the unique vector of equilibrium log-prices at a given vector of (inverse) log-productivity shocks $x \in \mathbb{R}^n$.

We denote the equilibrium aggregate log-output at such a vector of log-prices by

$$\Upsilon(x) = \max_{C \in \mathbb{R}_+^n} \ln U(C) \quad \text{s.t.} \quad \sum_{i=1}^n C_i \exp(\Pi_i(x)) \leq 1. \quad (15)$$

It is standard to show that:⁵

$$\Upsilon(x) = - \sum_{i=1}^n \xi_i \Pi_i(x). \quad (16)$$

While equilibrium prices and *aggregate* output are unique, there is not necessarily a unique input-output network. We characterize this stronger uniqueness property in Proposition 6.

3.4 Example Economies

We conclude this section by introducing four classes of example economies that fall within our general framework and to which we will often turn to exemplify our general results.

Example 1 (Constant Elasticity of Substitution). A standard specification of production in which the input-output network depends on equilibrium prices is the CES structure:

$$G_i(Q_i) = \left(\sum_{j=1}^n \alpha_{ij}^{\frac{1}{\sigma_i}} Q_{ij}^{\frac{\sigma_i-1}{\sigma_i}} \right)^{\frac{\sigma_i}{\sigma_i-1}} \quad (17)$$

for some vector of weights $\alpha_i \in \Delta$ and elasticity of substitution $\sigma_i \in \bar{\mathbb{R}}_+$. As is well-known, $\sigma_i = 0$ corresponds to the Leontief production function $G_i(Q_i) = \min_{j: \alpha_{ij} > 0} \left\{ \frac{Q_{ij}}{\alpha_{ij}} \right\}$, $\sigma_i = \infty$

⁴As T is monotone and translation invariant (see Proposition 1), Blackwell's sufficient conditions imply that the fixed point operator is a contraction.

⁵With more general preferences, the GDP formula would instead be a non-linear aggregate of prices. Under positive homogeneity of U , Proposition 1 can be applied directly to provide a general formula for GDP.

corresponds to the perfect substitutes production function $G_i(Q_i) = \sum_{j:\alpha_{ij}>0} Q_{ij}$, and $\sigma_i = 1$ corresponds to a Cobb-Douglas production function $G_i(Q_i) = \prod_{j=1}^n (Q_{ij}/\alpha_{ij})^{\alpha_{ij}}$. For CES technology, the log-cost function is given in general by:

$$T_i(p) = \frac{1}{1 - \sigma_i} \ln \left(\sum_{j=1}^n \alpha_{ij} \exp((1 - \sigma_i)p_j) \right). \quad (18)$$

Existing work has characterized the second-order local implications of shocks with nested CES (Baqaee and Farhi, 2019) and the asymptotic effects with large shocks (Dew-Becker, 2023). $\triangle$

Example 2 (Endogenous Cobb-Douglas). Recent work that studies endogenous production networks has considered models in which each sector $i \in [n]$ chooses its Cobb-Douglas input shares to economize on costs at the opportunity cost of perhaps suffering reduced productivity:

$$G_i(Q_i) = \max_{\alpha_i \in C_i} A_i(\alpha_i) \prod_{j=1}^n \left(\frac{Q_{ij}}{\alpha_{ij}} \right)^{\alpha_{ij}} \quad (19)$$

for some upper semicontinuous and log-concave productivity shifter $A_i : C_i \rightarrow [0, 1]$ where $C_i \subseteq \Delta$ is nonempty, compact, and convex, and $\max_{\alpha_i \in C_i} A_i(\alpha_i) = 1$. We interpret $A_i(\alpha_i)$ as the productivity of the sector when it operates using some feasible input shares $\alpha_i \in C_i$. The closer $A_i(\alpha_i)$ is to 1, the cheaper it is (in terms of lost productivity) for sector i to use technology α_i . The technologies $\alpha_i \notin C_i$ are the infeasible ones for sector i . In this case:

$$T_i(p) = \min_{\alpha_i \in C_i} \left\{ \sum_{j=1}^n \alpha_{ij} p_j + \ln \left(\frac{1}{A_i(\alpha_i)} \right) \right\} \quad (20)$$

Influential papers in the literature have studied particular cases of this class. Acemoglu and Azar (2020) consider as a leading application a specification in which productivity depends only on the identity of the suppliers chosen by each sector, but their input shares are fixed. Kopytov et al. (2024) study a specification in which $A_i(\alpha_i)$ is smooth and strictly log-concave. $\triangle$

Example 3 (Task-Based Production). Recent work that studies the substitutability of different factors of production has studied task-based models in which output is produced by completing a set of different tasks and each task has its own production function (see *e.g.*, Acemoglu and Restrepo, 2018). Inspired by this, we consider an economy in which the set of tasks for sector i is a nonempty, compact, and convex set $C_i \subseteq \Delta$. Output is only produced if each task in this set is completed. That is, each sector has a Leontief production function over tasks. The output of each task is itself given by a Cobb-Douglas production function with weights $\alpha_i \in C_i$ and a productivity shifter $A_i : C_i \rightarrow [1, \infty]$ with $\min_{\alpha_i \in C_i} A_i(\alpha_i) = 1$ and $C_i = \text{cl}\{\alpha_i \in \Delta : A_i(\alpha_i) < \infty\}$ that is lower semicontinuous and entropy log-convex, *i.e.*,

$w_i \mapsto \ln A_i(w_i) - \sum_{j=1}^n w_{ij} \ln w_{ij}$ is convex. Thus, the production function is given by:

$$G_i(Q_i) = \min_{\alpha_i \in C_i} A_i(\alpha_i) \prod_{j=1}^n \left(\frac{Q_{ij}}{\alpha_{ij}} \right)^{\alpha_{ij}}. \quad (21)$$

That is, the sector must acquire a vector of inputs Q_i that produces a unit of all the tasks in C_i . With this, the tasks such that $A_i(\alpha_i)$ is high are those for which the constraint is less binding, while the tasks such that $A_i(\alpha_i)$ is close to one are bottlenecks. In this case, the log-cost operator is given by that of Equation 20, with the min replaced by a max. $\triangle$

Example 4 (Endogenous CES). Our general framework allows us to study a new class of CES economies in which input weights and elasticities are both chosen endogenously:

$$G_i(Q_i) = \max_{\alpha_i \in \Delta, \sigma_i \in [0, \bar{\sigma}_i]} A_i(\alpha_i, \sigma_i) \left(\sum_{j=1}^n \alpha_{ij}^{\frac{1}{\sigma_i}} Q_{ij}^{\frac{\sigma_i-1}{\sigma_i}} \right)^{\frac{\sigma_i}{\sigma_i-1}} \quad (22)$$

where $\bar{\sigma}_i \in \mathbb{R}_+$ indexes the maximum possible degree of substitution and an upper-semicontinuous $A_i : \Delta \times \mathbb{R}_+ \rightarrow [0, 1]$ such that $\max_{\alpha_i \in \Delta, \sigma_i \in [0, \bar{\sigma}_i]} A_i(\alpha_i, \sigma_i) = 1$ represents the productivity of producing using different input weights and substitution elasticities. In this case:

$$T_i(p) = \min_{\alpha_i \in \Delta, \sigma_i \in [0, \bar{\sigma}_i]} \left\{ \frac{1}{1 - \sigma_i} \ln \left(\sum_{j=1}^n \alpha_{ij} \exp((1 - \sigma_i)p_j) \right) + \ln \left(\frac{1}{A_i(\alpha_i, \sigma_i)} \right) \right\}$$

A natural sub-class of the endogenous CES family emerges if we assume that sector $i \in [n]$ is endowed with some initial input weights α_i^0 and elasticity of substitution σ_i^0 and it can suffer an “adjustment cost” in productivity-equivalent units that is given by:

$$A_i(\alpha_i, \sigma_i) = \exp \left(-D(\alpha_i \| \alpha_i^0) - d(\sigma_i \| \sigma_i^0) \right) \quad (23)$$

where $D : \Delta \times \Delta \rightarrow \bar{\mathbb{R}}_+$ is a divergence between α_i and α_i^0 and $d : \bar{\mathbb{R}}_+^2 \rightarrow \bar{\mathbb{R}}_+$ is a divergence between σ_i and σ_i^0 . This specification captures the intuitive idea that if a sector tries to produce using a technology that is different from where it started, then there is an adjustment cost. $\triangle$

4 A Characterization of Equilibrium Prices and Output

In this section, we characterize the robust properties of equilibrium prices and output in terms of primitive production technologies. A key step in proving this result is showing that an arbitrary economy can be represented as one in which each sector chooses among Leontief technologies with different input weights and productivities (as in Koopmans, 1951).

4.1 A Min-Max Characterization of Log-Cost Functions

We begin our analysis by characterizing log-cost functions. To do this, we define some standard functional properties. A function $T_i : \mathbb{R}^n \rightarrow \mathbb{R}$ is *translation invariant* if $T_i(p + he) = T_i(p) + h$ for all $p \in \mathbb{R}^n$ and $h \in \mathbb{R}$ where $e = (1, \dots, 1)$, *normalized* if $T_i(0) = 0$, and *monotone* if $T_i(p') \geq T_i(p)$ for all $p' \geq p$. Similarly, an operator from $\mathbb{R}^n$ to itself satisfies any of the aforementioned properties if each of its component functions satisfies it. We also define the relative entropy $R : \Delta \times \Delta \rightarrow \overline{\mathbb{R}}_+$:

$$R(w||\tilde{w}) = \begin{cases} \sum_{j=1}^n w_j \ln \left(\frac{w_j}{\tilde{w}_j} \right) & \text{if } w \ll \tilde{w}, \\ \infty & \text{otherwise,} \end{cases} \quad (24)$$

where $w \ll \tilde{w}$ means that w is absolutely continuous with respect to $\tilde{w}$.⁶ We have that:

Proposition 1. *Fix $T_i : \mathbb{R}^n \rightarrow \mathbb{R}$. The following are equivalent:*

- (i) *There exists a regular production function G_i that induces T_i as the log-cost function.*
- (ii) *T_i is translation invariant, normalized, and $K_{T_i} := \exp \circ T_i \circ \ln$ is quasiconcave.*

Moreover, T_i is monotone and G_i and T_i are related through:

$$T_i(p) = \min_{\tilde{w} \in \Delta} \max_{w \in \Delta} \left\{ \sum_{j=1}^n w_j p_j - R(w||\tilde{w}) + \ln \left(\frac{1}{G_i(\tilde{w})} \right) \right\} \quad (25)$$

The translation invariance property of T follows from constant returns-to-scale, while the requirement that K_T is quasiconcave ensures that T is consistent with a valid cost minimization problem. We derive the representation of T by first showing that the cost function K corresponding to any production function G can always be written as:

$$K_i(P) = \min_{\tilde{w} \in \Delta} \frac{\sum_{j=1}^n \tilde{w}_j P_j}{G_i(\tilde{w})} \quad (26)$$

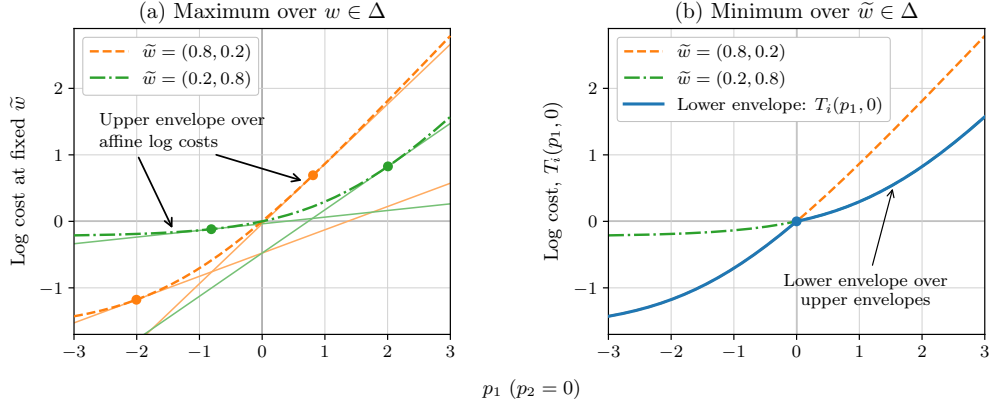
More evocatively, we observe that this is the same cost function as the one that would obtain if the production function were given by the following endogenous Leontief production function:

$$G_i^*(Q) = \max_{\tilde{w} \in \Delta} \left\{ G_i(\tilde{w}) \min_{j \in [n]: \tilde{w}_j > 0} \frac{Q_j}{\tilde{w}_j} \right\}$$

That is, it is *as if* any production function corresponds to choosing a Leontief technology with input weights $\tilde{w} \in \Delta$ and corresponding endogenous productivity $G_i(\tilde{w})$. Intuitively, the

⁶We adopt the convention that, for all $\tilde{w}, w \in \Delta$ such that $G_i(\tilde{w}) = 0$, we have $\ln \left(\frac{1}{G_i(\tilde{w})} \right) - R(w||\tilde{w}) = \infty$.

Figure 4: Illustration of the Log-Cost Formula from Proposition 1



Notes: Panel (a) of this figure plots the Leontief log cost function for two fixed Leontief weights $\tilde{w}$ as the dashed orange and green lines, with the pale green and orange lines corresponding to affine log cost functions that support the upper envelope as $w \in \Delta$ is maximized and dots denoting the points of tangency. Panel (b) plots the same orange and green lines from panel (a) and shows the log cost function as a function of p_1 (evaluated at $p_2 = 0$) for a sector that can choose between two specific Leontief technologies (as in Section 2).

resulting technology arises from the trade-off between choosing cheap inputs and selecting an input mix at which productivity is high.

In turn, this endogenous Leontief representation can be directly connected with the models of endogenous production technologies recently analyzed in the literature (Acemoglu and Azar, 2020; Kopytov et al., 2024). These papers consider production functions that can be obtained as upper envelopes over technologies of smooth production functions, usually Cobb-Douglas, with technology-dependent productivity shifters. This result shows that there is no loss of generality in making this assumption, so long as the base production technology is Leontief.⁷

To understand the final formula for T_i , we rewrite Equation 26 by taking logarithms:

$$T_i(p) = \min_{\tilde{w} \in \Delta} \left\{ \ln \left(\sum_{j=1}^n \tilde{w}_j \exp(p_j) \right) + \ln \left(\frac{1}{G_i(\tilde{w})} \right) \right\} \quad (27)$$

and then leveraging the Donsker-Varadhan formula to express the first term as:

$$\ln \left(\sum_{j=1}^n \tilde{w}_j \exp(p_j) \right) = \max_{w \in \Delta} \left\{ \sum_{j=1}^n w_j p_j - R(w || \tilde{w}) \right\} \quad (28)$$

which then directly yields Equation 25. The economic interpretation of this formula is that

⁷This may be reminiscent of Houthakker (1955), who shows that aggregation of Leontief technologies with fixed inputs and heterogeneous input-specific productivity can yield an aggregate Cobb-Douglas technology. We emphasize that our result is microeconomic, rather than arising from aggregating heterogeneous technologies.

the Leontief technology with weights $\tilde{w}$ is the worst possible Cobb-Douglas technology with a productivity gain that depends on the relative entropy between the chosen Cobb-Douglas weights w and the Leontief weights $\tilde{w}$.

The key mathematical upshot of this representation is that T_i is the min-max of an *affine* function of log-prices. We provide a graphical intuition for this fact in Figure 4. In panel (a), we illustrate how the Leontief log-cost function can be obtained as an upper envelope of affine log-cost functions (Equation 28). In panel (b), we show in the endogenous Leontief context from Section 2 (with two Leontief technologies) how the ultimate log-cost function is obtained as a lower envelope of these upper envelopes (Equation 27).

4.2 Equilibrium Price Characterization

We now characterize equilibrium prices. Let $\mathcal{W} = \{W \in \mathbb{R}_+^{n \times n} : \forall i \in [n], w_i \in \Delta\}$ be the set of $n \times n$ row-stochastic matrices, which can be interpreted as the set of all possible input-output matrices. Given an input-output matrix $W \in \mathcal{W}$, we define the Leontief inverse:

$$\Omega_W = (1 - \beta)(I - \beta W)^{-1} \quad (29)$$

It is well known (see *e.g.*, Acemoglu et al., 2012) that in a Cobb-Douglas economy with fixed input shares W , the equilibrium price vector is given by $\Pi(x) = \Omega_W x$. That is, Ω_W uniquely pins down how log-productivity shocks propagate in equilibrium. To generalize this, we also define the production-wedge operator $\tau^G : \mathcal{W} \times \mathcal{W} \rightarrow \mathbb{R}^n$ as:

$$\tau_i^G(W \| \tilde{W}) = \frac{\beta}{1 - \beta} \left[\ln \left(\frac{1}{G_i(\tilde{w}_i)} \right) - R(w_i \| \tilde{w}_i) \right] \quad \forall i \in [n] \quad (30)$$

The production wedge is simply a rescaling of the additive productivity distortion term in the affine min-max representation of T_i (see Equation 25).⁸

With these objects in hand, we state our characterization of equilibrium prices:

Theorem 1 (Equilibrium Prices). *Fix $\Pi : \mathbb{R}^n \rightarrow \mathbb{R}^n$. The following are equivalent:*

- (i) *There exists a profile of regular production functions $G = (G_i)_{i=1}^n$ that induces Π as the equilibrium log-price operator.*
- (ii) *Π is translation invariant, normalized, bijective, and $\tilde{K}_\Pi = \exp \circ \left(\frac{I - (1 - \beta)\Pi^{-1}}{\beta} \right) \circ \ln$ is quasiconcave.*

Moreover, Π is monotone and G and Π are related through:

$$\Pi(x) = \min_{\tilde{W} \in \mathcal{W}} \max_{W \in \mathcal{W}} \left\{ \Omega_W \left(x + \tau^G(W \| \tilde{W}) \right) \right\} \quad (31)$$

⁸With the convention that for all $\tilde{W}, W \in \mathcal{W}$, if $\exists i \in [n]$ such that $G_i(\tilde{w}_i) = 0$, then $\tau_j^G(W \| \tilde{W}) = \infty \forall j \in [n]$.

The first part of this result provides a complete characterization of the possible equilibrium log-price operators in terms of functional properties, mirroring the microeconomic characterization of possible log-cost functions from Proposition 1. While many such equilibrium price maps are possible, in the spirit of classical “anything goes” theorems (Sonnenschein, 1973; Mantel, 1974; Debreu, 1974), the production network structure of the economy does impose substantive restrictions. First, equilibrium log-prices must be translation invariant: proportional shifts in sectoral productivities must leave relative prices unchanged. Second, no two vectors of sectoral productivities can yield the same equilibrium prices and every vector of equilibrium prices can be rationalized by some vector of sectoral productivities. One implication of this property is that observation of relative prices without knowledge of productivities reveals no information about the structure of the economy. Finally, price maps must be consistent with cost minimization in the sense that the unique potential cost functions that rationalize the observed prices $\tilde{K}_\Pi$ are in fact valid cost functions, *i.e.*, they are quasiconcave.

The second part of Theorem 1 provides an explicit formula mapping microeconomic technology to equilibrium prices. In particular, the right-hand side of Equation 31 is entirely *primitive*: it depends only on the production functions and not on any equilibrium objects. To interpret the formula, fix $\tilde{W}$ and W that respectively are minimizers and maximizers in Equation 31. We then have that $\Pi(x) = \Omega_W(x + \tau^G(W\|\tilde{W}))$. It is as if the initial adverse productivity shock x is further increased by the production wedge $\tau^G(W\|\tilde{W})$, and then this composite productivity shock propagates according to the Leontief inverse of the input-output matrix W . This is the formula for equilibrium prices in a Cobb-Douglas economy with input-output matrix W and inverse log-productivity vector $x + \tau^G(W\|\tilde{W})$.

We now provide intuition for the fact that equilibrium log-prices in the non-linear economy correspond to those of a distorted linear (Cobb-Douglas) economy. Applying the endogenous Leontief representation of Proposition 1, the fixed point equation for equilibrium prices becomes:

$$p_i = (1 - \beta)x_i + \beta \min_{\tilde{w}_i \in \Delta} \max_{w_i \in \Delta} \left\{ \sum_{j=1}^n w_{ij} p_j - R(w_i\|\tilde{w}_i) + \ln \left(\frac{1}{G_i(\tilde{w}_i)} \right) \right\}, \quad \forall i \in [n] \quad (32)$$

Thus, stacking these equations and using the definition of the production-wedge operator:

$$p = (1 - \beta)x + \beta \min_{\tilde{W} \in \mathcal{W}} \max_{W \in \mathcal{W}} \left\{ Wp + \frac{1 - \beta}{\beta} \tau^G(W\|\tilde{W}) \right\} \quad (33)$$

To understand why we can view row-by-row min-maximization over vectors as vectorial min-maximization over matrices, observe that each sector chooses its own technology, taking the prices as given. We then proceed by doing something seemingly naive. Suppose that we fixed a given pair of input-output matrices $(\tilde{W}, W)$. The resulting fixed point equation is simply a

linear system in p and can be exactly inverted to yield:

$$p = \Omega_W(x + \tau^G(W||\tilde{W})) \quad (34)$$

The key step of the proof is then to show that we can apply the min-max structure to this solution and obtain the solution that we would have obtained if we applied the min-max and then computed the fixed point. That is, the operations of taking the fixed point and the min-max can be performed in either order and the answer is the same.⁹ As a byproduct, this result guarantees the existence of a pair of input-output networks W and $\tilde{W}$ that respectively and *simultaneously* maximize and minimize the price of *every* sector.

With CES production functions, the production wedge also takes on a particularly simple form, which is related to common statistical measures of distance:

Example 1 (continuing from p. 10). If G_i is a CES production function with weights $\alpha_i \in \Delta$ and $\sigma_i \in \overline{\mathbb{R}}_+$, then the wedge is proportional to the Rényi divergence with parameter $\frac{1}{\sigma_i}$ between α_i and $\tilde{w}_i$ and the relative entropy between w_i and $\tilde{w}_i$:

$$\tau_i^G(W||\tilde{W}) = \frac{\beta}{1-\beta} \left[D_{\sigma_i^{-1}}(\alpha_i||\tilde{w}_i) - R(w_i||\tilde{w}_i) \right] \quad (35)$$

where D_t is the Rényi divergence with parameter $t \in \overline{\mathbb{R}}$. In the Cobb-Douglas case in which $\sigma_i = 1$, the Rényi divergence becomes the relative entropy. $\triangle$

We conclude by highlighting how Theorem 1 covers distorted economies, implies a formula for equilibrium GDP, and relates to the decision theory literature.

Remark 1 (The Price Formula in Distorted Economies). Our baseline model assumes that there is perfect competition. However, our results characterizing prices extend immediately to settings with fixed markups or other productivity distortions (such as taxes or subsidies). To be more specific, if each sector has markup $\mu_i \in \mathbb{R}_{++}$ (as in, *e.g.*, Baqaee and Farhi, 2020; Bigio and La'O, 2020), then redefining $\tilde{Z}_i = Z_i/\mu_i$ and applying Theorem 1 immediately yields the formula for equilibrium log-prices in distorted economies. $\triangle$

Remark 2 (From Prices to GDP). With prices derived, equilibrium GDP follows simply by solving the household's utility maximization problem given the equilibrium prices. As we showed earlier, log-GDP is given by $\Upsilon(x) = -\sum_{i=1}^n \xi_i \Pi_i(x)$. It follows immediately from Theorem 1 that the corresponding equilibrium log-output is given by:

$$\Upsilon(x) = \max_{\tilde{W} \in \mathcal{W}} \min_{W \in \mathcal{W}} \left\{ -\xi' \Omega_W \left(x + \tau^G(W||\tilde{W}) \right) \right\} \quad (36)$$

⁹By way of analogy, this ability to interchange optimality and fixed points appears regularly in economics via the Bellman theorem: the value of a dynamic optimization problem (under standard conditions) is the same as the fixed point of a Bellman operator.

Intuitively, as we are solving all n min-max problems simultaneously, the expenditure weights pass through the min-max structure. $\triangle$

Remark 3 (Relationship to Decision Theory Literature). We observe that the fixed point operators for equilibrium log-prices described by Lemma 1 under the restrictions of monotonicity, normalization, and translation invariance (which are implied by Proposition 1) coincide with non-linear operators that commonly arise in decision theoretic analyses (see *e.g.*, [Maccheroni et al., 2006](#); [Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio, 2011](#); [Cerreia-Vioglio, Maccheroni, Marinacci, and Rustichini, 2014](#)) and have been applied by [Cerreia-Vioglio et al. \(2026\)](#) in broader contexts. [Cerreia-Vioglio et al. \(2026\)](#) characterize the limit as $\beta \rightarrow 1$ (when appropriately renormalized) of such fixed points. We characterize equilibrium prices in general—thus allowing for a labor share of production that is far from zero—leveraging the additional concavity structure that Proposition 1 implies. $\triangle$

4.3 Illustrating The Formula in Two-Sector Economies

We now illustrate Theorem 1 in a setting with one upstream sector and one downstream sector, like the one analyzed in Section 2. As before, assume that the production function of the upstream sector is given by $Y_U = 2Z_U Q_{U,0}^{\frac{1}{2}} Q_{U,U}^{\frac{1}{2}}$. Now, instead of the double-Leontief form of the initial example, we generalize the downstream production function to take the form $Y_D = 2Z_D Q_{D,0}^{\frac{1}{2}} G_D(Q_{D,D}, Q_{D,U})^{\frac{1}{2}}$, where G_D is regular. For simplicity, we assume that the household consumes only downstream output and set $Z_D = 1$.

We now use Theorem 1 to understand the propagation of upstream productivity shocks into real GDP. With $x_U = -2 \ln Z_U$, the unique equilibrium log prices are given by:

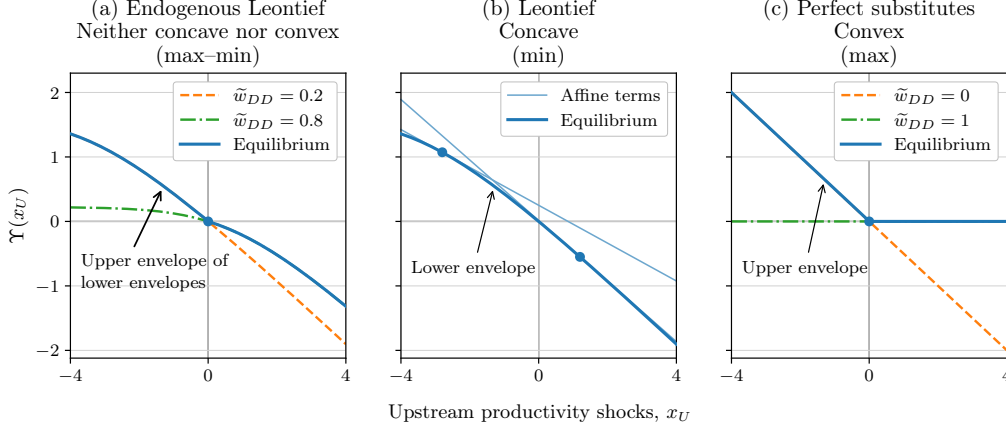
$$\begin{pmatrix} p_U \\ p_D \end{pmatrix} = \min_{\tilde{W} \in \mathcal{W}} \max_{W \in \mathcal{W}} \left\{ \begin{pmatrix} \frac{2-w_{DD}}{3-w_{UU}-w_{DD}} & \frac{1-w_{UU}}{3-w_{UU}-w_{DD}} \\ \frac{1-w_{DD}}{3-w_{UU}-w_{DD}} & \frac{2-w_{UU}}{3-w_{UU}-w_{DD}} \end{pmatrix} \begin{pmatrix} x_U + \ln \left(\frac{1}{\tilde{w}_{UU}} \right) - R(w_U || \tilde{w}_U) \\ \ln \left(\frac{1}{G_D(\tilde{w}_D)} \right) - R(w_D || \tilde{w}_D) \end{pmatrix} \right\} \quad (37)$$

where we have simplified the Leontief inverse. We begin by inspecting the first row of this problem. First, observe that the minimizer can guarantee a value of x_U by setting $\tilde{w}_{UU} = 1$ (as the maximizer must then set $w_{UU} = 1$). Second, observe that the maximizer can also guarantee a value of x_U by setting $w_{UU} = 1$ (as the production wedge cancels to zero at this point). Thus, we have that $p_U = x_U$. We now turn to the second row, which allows us to express log real GDP $\Upsilon(x_U) = -p_D$ as:

$$\Upsilon(x_U) = \max_{\tilde{w}_{DD} \in [0,1]} \min_{w_{DD} \in [0,1]} \left\{ -\frac{1}{2-w_{DD}} \left((1-w_{DD})x_U + \hat{\tau}_D(w_{DD}, \tilde{w}_{DD}) \right) \right\} \quad (38)$$

where $\hat{\tau}_D(s, \tilde{s}) = -\ln G_D(\tilde{s}, 1-s) - R((s, 1-s) || (\tilde{s}, 1-\tilde{s}))$ is the production wedge of the

Figure 5: Illustration of the Min-Max Formula from Theorem 1 in Example Economies



Notes: This figure plots equilibrium log GDP in Examples 5, 6, and 7. Panel (a) plots hypothetical log GDP with fixed Leontief technologies $\omega = 0.2, 0.8$ as orange and green lines respectively, with their upper envelope corresponding to equilibrium log GDP as per Equation 41. Panel (b) plots equilibrium log GDP in a Leontief economy as the solid blue line as per Equation 39, with pale blue lines corresponding to two supporting affine log GDP functions with fixed w_{DD} , and the dots denoting the point of tangency of these affine functions with the lower envelope. Panel (c) plots hypothetical log GDP under perfect substitutes technologies in which downstream uses no downstream inputs (orange line) and downstream only uses downstream inputs (green line) with the upper envelope of these (blue line) corresponding to the true equilibrium log GDP as per Equation 40.

downstream sector. We now unpack the intuition behind this formula and the role of the max and min through a series of examples.

Example 5 (Perfect Complements). Suppose that the downstream sector operates a Leontief technology $G_D(Q_{D,D}, Q_{D,U}) = \min\{Q_{D,D}/\alpha, Q_{D,U}/(1-\alpha)\}$ for $\alpha \in (0, 1)$. In the formula above, it is simple to verify that $\tilde{w}_{DD} = \alpha$ is optimal and hence that:

$$\Upsilon^{\text{Leontief}}(x_U) = \min_{w_{DD} \in [0,1]} \left\{ -\frac{(1-w_{DD})x_U - R((w_{DD}, 1-w_{DD}) || (\alpha, 1-\alpha))}{2-w_{DD}} \right\} \quad (39)$$

Thus, log GDP is a minimum of an *affine* function of x_U and is hence concave in upstream productivity shocks. We visualize this in panel (b) of Figure 5. Thus, in Leontief economies, which are intuitively the most fragile to shocks, the force of the minimum dominates. $\triangle$

Example 6 (Perfect Substitutes). To consider the polar opposite case, now suppose that the downstream sector operates a perfect substitutes production technology $G_D(Q_{D,D}, Q_{D,U}) = Q_{D,D} + Q_{D,U}$. Again, in the GDP formula, one can verify that $w_{DD} = \tilde{w}_{DD}$ and hence that:

$$\Upsilon^{\text{Substitutes}}(x_U) = \max_{\tilde{w}_{DD} \in [0,1]} \left\{ -\frac{(1-\tilde{w}_{DD})x_U}{2-\tilde{w}_{DD}} \right\} \quad (40)$$

And so log GDP is a maximum of an affine function of x_U and is therefore convex in upstream productivity shocks. We visualize this in panel (c) of Figure 5. Hence, in economies with perfect substitutes, which are intuitively the most resilient to shocks, the force of the maximum dominates. $\triangle$

In general, the forces of substitution and complementarity do not give rise to either a globally concave solution (as in the Leontief case) or a globally convex solution (as in the perfect substitutes case), as we demonstrate in the following two examples:

Example 7 (Choice of Leontiefs). As in Section 2, the downstream sector can operate two Leontief technologies, $G_D(Q_{D,D}, Q_{D,U}) = \max_{\omega \in \{\underline{\omega}, \bar{\omega}\}} \min \{Q_{D,D}/\omega, Q_{D,U}/(1-\omega)\}$. In this context, equilibrium log GDP is given by the max-min formula:

$$\Upsilon(x_U) = \max_{\omega \in \{\underline{\omega}, \bar{\omega}\}} \min_{w_{DD} \in [0,1]} \frac{-(1-w_{DD})x_U + R((w_{DD}, 1-w_{DD}) \parallel (\omega, 1-\omega))}{2-w_{DD}} \quad (41)$$

and log GDP is a max-min of an affine function of x_U and is neither concave nor convex in upstream productivity shocks, as we saw in the initial example. We visualize this in panel (a) of Figure 5. $\triangle$

Example 4 (continuing from p. 12). We finally consider an example in which technology can change smoothly. The downstream sector operates an endogenous CES technology of the form:

$$G_D(Q_{D,D}, Q_{D,U}) = \max_{\alpha \in [0,1], \sigma \in [0,1]} A(\alpha, \sigma) \left(\alpha^{\frac{1}{\sigma}} Q_{D,D}^{\frac{\sigma-1}{\sigma}} + (1-\alpha)^{\frac{1}{\sigma}} Q_{D,U}^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \quad (42)$$

where α corresponds to the weight of downstream inputs in production and σ is the elasticity of substitution between upstream and downstream inputs. These parameters are endogenously chosen to maximize profits and changing the parameters away from some baseline values $\alpha_0 \in [0, 1]$ and $\sigma_0 \in [0, 1]$ entails an adjustment cost in productivity units of:

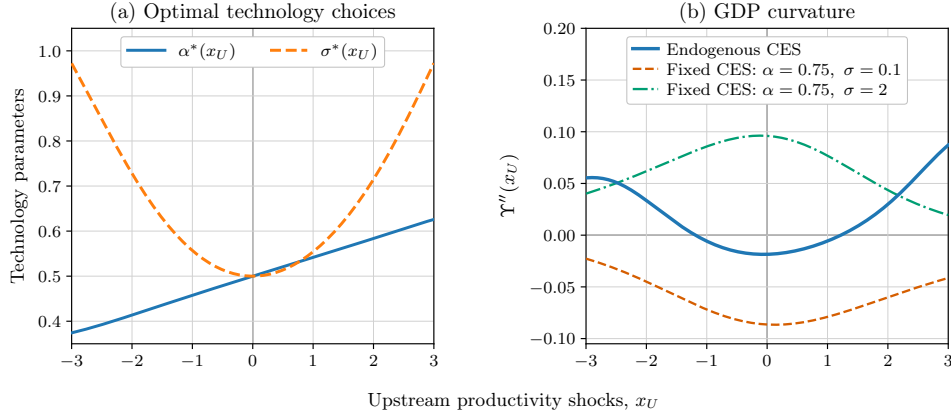
$$A(\alpha, \sigma) = \exp \left(-\frac{\kappa_\alpha}{2} (\alpha - \alpha_0)^2 - \frac{\kappa_\sigma}{2} (\sigma - \sigma_0)^2 \right) \quad (43)$$

where $\kappa_\alpha, \kappa_\sigma > 0$ denote the magnitude of the adjustment costs of changing input weights and substitutability, respectively. Intuitively, these parameters capture the difficulty in adjusting input needs and substitutability in the face of shocks. It is natural to imagine that they are very large in the short-run, while being lower in the long-run. Theorem 1 implies that:

$$\Upsilon(x_U) = \max_{\alpha \in [0,1], \sigma \in [0,1]} \min_{w_{DD} \in [0,1]} \frac{-(1-w_{DD})x_U + \ln A(\alpha, \sigma) + \frac{R((w_{DD}, 1-w_{DD}) \parallel (\alpha, 1-\alpha))}{1-\sigma}}{2-w_{DD}} \quad (44)$$

and log GDP is once again a max-min of an affine function of x_U .

Figure 6: Adaptation and GDP Curvature in an Endogenous CES Economy



Notes: This figure plots optimal technology choices for α and σ as a function of upstream productivity shocks x_U , $\alpha^*(x_U)$ and $\sigma^*(x_U)$ respectively in panel (a) and the second derivative of equilibrium log GDP in panel (b) as a function of upstream productivity shocks. In panel (b), we also plot the curvature of GDP of two fixed CES technologies (where α and σ are exogenously fixed). Parameters: $\alpha_0 = \frac{1}{2}$, $\sigma_0 = \frac{1}{2}$, $\kappa_\alpha = 16$, $\kappa_\sigma = 1$.

We visualize the optimal technology choices alongside the lack of global concavity or convexity of GDP in Figure 6. In panel (a) we plot the optimal choices of α and σ as upstream productivity varies. As is intuitive, when upstream becomes less productive (x_U increases), downstream optimally increases α to become less reliant on upstream goods. Additionally, when upstream either becomes more productive or less productive, downstream optimally increases σ to be able to better take advantage of the relatively cheaper input. In panel (b), we plot the implications of these technology adaptations for the curvature of GDP, $\gamma''(x_U)$. In this example, for small productivity shocks, GDP is concave as the sector's adaptation is minimal and $\sigma < 1$. For large shocks, GDP is convex—despite the fact that the optimal choice of σ is always smaller than one—as the sector changes its technology more rapidly in the face of greater incentives to do so. We contrast this with the case of exogenous CES technology, in which the response is always convex for $\sigma > 1$ and always concave for $\sigma < 1$ (panel (b)). $\triangle$

Thus, in general, neither the minimum nor the maximum dominates globally, and the two interact to determine the response of prices and GDP to shocks. In the following section, we proceed to formally define notions of resilience and fragility and we show that these concepts map formally to the forces of the minimum and the maximum in the general representation.

5 Application I: Resilience *vs.* Fragility

We now apply our general representation of equilibrium prices and formalize the economic sense in which the general min-max formula encodes the dueling forces of resilience and fragility.

5.1 Defining Resilience and Fragility

To define notions of resilience and fragility, we ask: when is it true that making shocks to productivity more dispersed has adverse economic consequences? Informally, if the consequences are adverse, we call the economy fragile, and if the consequences are beneficial, we call the economy resilient. While this is not the only reasonable definition of fragility and resilience, it corresponds to the popular notion advanced by Taleb (2013): “Fragility can be defined as an accelerating sensitivity to a harmful stressor: this response plots as a concave curve and mathematically culminates in more harm than benefit from random events. Antifragility is the opposite, producing a convex response that leads to more benefit than harm.”

Definition 2 (Resilience and Fragility). *Sector i is resilient (fragile) if T_i is concave (convex). The economy is resilient (fragile) if Π is concave (convex).*

To motivate this definition, assume that the log-prices faced by sector i have an initial distribution π over $\mathbb{R}^n$. The expected log-cost of sector i is then $\mathbb{E}_\pi[T_i]$. We are interested in the average percentage effect of the percentage variation in input prices on costs. More specifically, consider a new distribution π' of log-prices, constructed by adding the original log-prices and a noise vector that is mean-zero and mean-independent with respect to the original log-prices.¹⁰ With this, sector i is resilient *if and only if* $\mathbb{E}_{\pi'}[T_i] \leq \mathbb{E}_\pi[T_i]$ for any such pair of distributions, *i.e.*, the new expected log-cost *is guaranteed* to be weakly below the original one. Thus, sector i takes advantage of the positive realizations of the new price shock more than it suffers from its negative realizations, so they at least secure the original log-cost on average. Conversely, sector i is fragile *if and only if* $\mathbb{E}_{\pi'}[T_i] \geq \mathbb{E}_\pi[T_i]$ for any such pair of distributions.

Thus, resilient sectors have lower log-costs in expectation when the volatility of log-prices rises, while the opposite happens for fragile sectors. We can interpret resilience and fragility of the economy analogously. An economy is resilient (fragile) if it benefits (suffers) in percentage on average from more volatility in percent productivity.

This notion of resilience and fragility is consistent with the intuitive idea that CES technologies are fragile (resilient) when the elasticity of substitution is smaller (larger) than one:

Example 1 (continuing from p. 10). A sector $i \in [n]$ with CES production technology with elasticity of substitution σ_i is resilient if and only if $\sigma_i \geq 1$ and fragile if and only if $\sigma_i \leq 1$. $\triangle$

5.2 Resilience

We now characterize resilience at the sector level and show that equilibrium prices preserve resilience. The next result shows that a sector is resilient if and only if its cost operator is

¹⁰Formally, π' is the distribution of the random vector $\hat{p}' = \hat{p} + \varepsilon$, where $\hat{p}$ is the original log-price vector with distribution π and ε is a random vector such that $\mathbb{E}[\varepsilon|p] = 0$ for π -almost every p .

generated by an endogenous Cobb-Douglas production function (as in Example 2).

Proposition 2. *Fix $i \in [n]$ with log-cost function T_i . The following are equivalent:*

- (i) *Sector i is resilient.*
- (ii) *T_i is generated by an endogenous Cobb-Douglas production function G_i^* :*

$$G_i^*(Q_i) = \max_{w_i \in C_i} \left\{ A_i(w_i) \prod_{j=1}^n \left(\frac{Q_{ij}}{w_{ij}} \right)^{w_{ij}} \right\} \quad (45)$$

for a nonempty, compact, and convex set $C_i \subseteq \Delta$ and production shifter $A_i : C_i \rightarrow [0, 1]$ that is upper semi-continuous, log-concave, and such that $\max_{w_i \in C_i} A_i(w_i) = 1$.

In this case, G_i^ is the unique regular and quasiconcave production function that generates T_i .*

One important implication of this result is that endogenous Cobb-Douglas production functions span the full scope of possibilities under resilience. Thus, insofar as one is interested in resilient technologies, this result implies that the approach of imposing endogenous Cobb-Douglas technology (as done in applications by Acemoglu and Azar, 2020; Kopytov et al., 2024) is not a restriction. However, this formulation rules out the possibility of fragility.

Example 1 (continuing from p. 10). Because CES production functions with elasticity $\sigma_i > 1$ induce a concave log-cost operator, it follows that they are cost-equivalent to some endogenous Cobb-Douglas production function. By inspection of the proof of Proposition 2, it is possible to show that the productivity shifter recovering the CES log-cost operator is $A_i(w_i) = \exp\left(-\frac{1}{\sigma_i-1}R(w_i||\alpha_i)\right)$ and $C_i = \{w_i \in \Delta : w_i \ll \alpha_i\}$, where $\alpha_i \in \Delta$ are the weights in the original CES production function. $\triangle$

The next result uses this representation and Theorem 1 to derive the formula for equilibrium prices in economies in which all sectors are resilient:

Corollary 1. *If every sector is resilient, then the economy is resilient and such that*

$$\Pi(x) = \min_{W \in \mathcal{W}_C} \left\{ \Omega_W(x + \tau^A(W)) \right\} \quad (46)$$

where $\mathcal{W}_C = \{W \in \mathcal{W} : w_i \in C_i, \forall i \in [n]\}$ and $\tau_i^A(W) = \frac{\beta}{1-\beta} \ln \left(\frac{1}{A_i(w_i)} \right)$ for all $i \in [n]$.

Thus, in resilient economies, the max in the general min-max representation of Theorem 1 disappears. This justifies our earlier informal interpretation of the min as inducing resilience. In this case, the production wedge also simplifies: G_i is replaced by A_i from the endogenous Cobb-Douglas representation.

5.3 Fragility

We now turn to the opposite question of when sectors and the aggregate economy are fragile. The following result shows that fragility is equivalent to a sector having a task-based production function with an entropy log-convex shifter A_i (as in Example 3).

Proposition 3. *Fix $i \in [n]$ with log-cost function T_i . The following are equivalent:*

- (i) *Sector i is fragile*
- (ii) *T_i is generated by a task-based production function G_i^* :*

$$G_i^*(Q_i) = \min_{w_i \in C_i} \left\{ A_i(w_i) \prod_{j=1}^n \left(\frac{Q_{ij}}{w_{ij}} \right)^{w_{ij}} \right\} \quad (47)$$

for a nonempty, compact, and convex set $C_i \subseteq \Delta$ and production shifter $A_i : C_i \rightarrow [1, \infty]$ that is lower semi-continuous, entropy log-convex, and satisfies $\min_{w_i \in C_i} A_i(w_i) = 1$.

In this case, G_i^ is the unique regular and quasiconcave production function that generates T_i .*

While the endogenous Cobb-Douglas production function that characterizes resilience is familiar in the literature, this task-based mirror is, to the best of our knowledge, new. Moreover, the shifter in the task-based production function is required to be entropy log-convex, a stronger property than log-convexity. This stronger requirement comes from the need to be consistent with the co-occurrence of T_i being convex while K_i is concave. Indeed, this requirement is so strong that convexity of T_i implies differentiability of T_i .¹¹ Intuitively, fragile sectors cannot change their production technique fast enough to generate kinks in T_i .

Example 1 (continuing from p. 10). Similarly to the resilient case, because CES production functions with elasticity $\sigma_i < 1$ induce a convex log-cost operator, it follows that they are cost-equivalent to some task-based production function. By inspection of the proof of Proposition 3, it is possible to show that the productivity shifter recovering the CES log-cost operator is $A_i(w_i) = \exp\left(\frac{1}{1-\sigma_i} R(w_i || \alpha_i)\right)$ and $C_i = \{w_i \in \Delta : w_i \ll \alpha_i\}$. $\triangle$

The next result leverages this representation and Theorem 1 to provide the equilibrium prices in economies in which all sectors are fragile:

Corollary 2. *If every sector is fragile, then the economy is fragile and such that*

$$\Pi(x) = \max_{W \in \mathcal{W}_C} \{ \Omega_W(x + \tau^A(W)) \} \quad (48)$$

¹¹The log-cost function of a fragile sector is Gateaux differentiable because its production shifter A_i is entropy log-convex, hence, strictly log-convex (see Bauschke, Borwein, and Combettes, 2001). More generally, one can combine Theorem 1 and this observation to prove that the equilibrium log-price operator of a fragile economy is always Gateaux differentiable.

Thus, just as resilience eliminated the max in the general min-max representation of Theorem 1, fragility eliminates the min in the representation. This completes our justification of the general max as corresponding to fragility and the min as corresponding to resilience. Moreover, as in the resilient case, the production wedge is substantially simpler than the general representation.

5.4 Resilience, Fragility, and the Ability to Abate Adverse Shocks

In this subsection, we show that our notions of resilience and fragility provide a useful diagnostic for the concept that a sector is (un)able to abate large shocks to upstream suppliers. Similarly to the analysis of Dew-Becker (2023), consider a pair of sectors $i, j \in [n]$ and suppose that there is a change in the price of sector j by an amount $\lambda > 0$ such that the input price vector changes according to $p \mapsto p + \lambda e_j$. We say that shocks to sector j can be *abated* by sector i if:¹²

$$\lim_{\lambda \rightarrow \infty} \frac{T_i(p + \lambda e_j) - T_i(p)}{\lambda} = 0 \quad (49)$$

Intuitively, a shock to j can be abated by sector i if, as the shock to j 's price becomes large, the ultimate effect on the costs of sector i becomes proportionately insignificant. To exemplify this definition, in the case in which sector i has a Cobb-Douglas production technology with weights $\alpha_i \in \Delta$, i can abate shocks to j if and only if $\alpha_{ij} = 0$.

To describe how resilience and fragility relate to the ability to abate shocks, consider first a resilient sector $i \in [n]$. By Proposition 2, such a sector can be represented as possessing an endogenous Cobb-Douglas production function with productivity shifter $A_i : C_i \rightarrow [0, 1]$. For such a production function, we define the set of feasible production technologies $C_i^R = \text{cl}\{w_i \in C_i : A_i(w_i) > 0\}$, *i.e.*, the closure of the set of all of the mixes of inputs such that the sector can generate production using that input mix. Now consider a fragile sector $i \in [n]$. By Proposition 3, such a sector can be represented via a task-based production function with productivity shifter $A_i : C_i \rightarrow [1, \infty]$. For such a production function, we define the set of necessary tasks $C_i^F = \{w_i \in C_i : A_i(w_i) < \infty\}$, *i.e.*, the set of tasks which must be completed in order to produce output. With these definitions in hand, we have that:

Proposition 4. *The following statements are true:*

1. *Suppose sector i is resilient. Sector i can abate shocks from sector j if and only if $w_{ij} = 0$ for some $w_i \in C_i^R$.*
2. *Suppose sector i is fragile. Sector i can abate shocks from sector j if and only if $w_{ij} = 0$ for all $w_i \in C_i^F$.*

¹²We observe that this property is independent of the initial p at which the object inside the limit is evaluated.

Thus, our notion of resilience captures the idea that a sector can abate adverse shocks so long as it is *possible* to adopt a technology that substitutes away from an adversely affected supplier. By contrast, our notion of fragility captures the idea that a sector can only abate shocks if *all* of the tasks that it performs do not require the outputs of the shocked supplier. Moreover, if a sector is fragile, it follows immediately that there always exists some sector such that shocks to that sector cannot be abated.

Example 1 (continuing from p. 10). Consider a sector i with a CES production technology. If sector i is resilient ($\sigma_i > 1$), then it can abate shocks to sector j if and only if $\alpha_{ij} < 1$. That is, it can abate shocks originating from a sector if and only if that sector is not its sole input supplier. If sector i is fragile ($\sigma_i < 1$), then it can abate shocks to sector j if and only if $\alpha_{ij} = 0$. That is, it can abate shocks from a sector only if it is not exposed to that sector. $\triangle$

5.5 Alternative Fragility Notions: Big Shocks and Elasticity

So far, we have considered a notion of fragility that corresponds to more dispersed shocks leading to worse outcomes. While this is a standard notion, there are other reasonable definitions of fragility, including that: (i) as shocks become bigger, their total adverse effect rises more than proportionately, and (ii) the elasticity of substitution is greater or smaller than one. In this section, we show how these notions are related.

Formally, we define resilience and fragility to big shocks as:

Definition 3 (Resilience and Fragility to Big Shocks). *Sector i is resilient (fragile) to big shocks if T_i is concave (convex) at zero. The economy is resilient (fragile) to big shocks if Π is concave (convex) at zero.*

We observe that T_i is convex at zero if and only if for all $\lambda \in (0, 1)$ and $p \in \mathbb{R}^n$:

$$T_i(p) \geq \frac{T_i(\lambda p)}{\lambda} \quad (50)$$

That is, the *total effect* of a shock in direction p —or $T_i(p)$ —is larger than the *average effect* for all scaled down versions of that shock— or $T_i(\lambda p)/\lambda$.

Moreover, we define elasticity as:

Definition 4 (Elasticity). *Sector i is elastic (inelastic) if T_i is supermodular (submodular).*

To understand why we use this definition of elasticity, we observe that when T_i is twice continuously differentiable, it collapses to the standard notion of elasticity from producer theory. More technically, given the corresponding cost function K_i , we define the Allen-Uzawa elasticity

of substitution as

$$\sigma_{jk}^{K_i}(P) = \frac{K_i(P) \frac{\partial^2 K_i(P)}{\partial P_j \partial P_k}}{\frac{\partial K_i(P)}{\partial P_j} \frac{\partial K_i(P)}{\partial P_k}} = 1 + \frac{\frac{\partial^2 T_i(p)}{\partial p_j \partial p_k}}{\frac{\partial T_i(p)}{\partial p_j} \frac{\partial T_i(p)}{\partial p_k}} \quad (51)$$

where the second equality follows by simple algebra. With this, supermodularity (submodularity) is equivalent to the elasticity of substitution being globally greater than one (smaller than one)—the standard notion of elastic (inelastic). Indeed, for a CES production function, this notion of elasticity corresponds to the standard condition that $\sigma_i \leq 1$. Our definition is more general in that it can be specified without assuming any differentiability property of T_i .

From this, we can see that elasticity encodes a notion of fragility to correlation. To understand this, suppose as before that the log-prices faced by sector i have distribution π on $\mathbb{R}^n$. A standard notion of a random vector with distribution π' being more correlated than another random vector with distribution π is that the distribution π' dominates π in the supermodular stochastic order (see [Shaked and Shanthikumar, 2007](#), for a discussion). Intuitively, if π' is more correlated than π , the expected value of a supermodular function will be larger under π' than π because the arguments are more likely to be *jointly* large and small.¹³ Thus, a sector being elastic is equivalent to being fragile to correlation.

The next result shows that these different notions of resilience and fragility can be ranked.

Proposition 5. *The following statements are true:*

1. *If a sector is inelastic (elastic), then it is fragile (resilient)*
2. *If a sector is fragile (resilient), then it is fragile (resilient) to big shocks*

Thus, while all sectors being elastic or inelastic is sufficient for the conclusion that the economy is resilient or fragile, it is not necessary. While these notions are equivalent with CES production technologies, away from this special case our analysis makes clear the distinct roles played by elasticity and resilience/fragility: elasticity determines the extent to which correlation is good or bad, while resilience and fragility determine the extent to which volatility is good or bad. As Theorem 1 and Corollaries 1 and 2 have shown that resilience and fragility characterize the min-max structure of equilibrium outcomes, our results indicate that elasticity (while more conventional) is less reflective of this general structure. The following nested CES example provides a particularly simple illustration of a sector that is fragile while not being inelastic:

Example 8 (Fragility without Inelasticity). Consider the nested CES production function with an outer CES nest with elasticity of substitution $\sigma \in (0, 1)$ and an inner Cobb-Douglas nest:

$$G(Q_1, Q_2, Q_3) = \left(\alpha^{\frac{1}{\sigma}} \left[\left(\frac{Q_1}{a} \right)^a \left(\frac{Q_2}{1-a} \right)^{1-a} \right]^{\frac{\sigma-1}{\sigma}} + (1-\alpha)^{\frac{1}{\sigma}} Q_3^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \quad (52)$$

¹³That is, for all supermodular $f : \mathbb{R}^n \rightarrow \mathbb{R}$, we have that $\mathbb{E}_{\pi'}[f(p)] \geq \mathbb{E}_{\pi}[f(p)]$.

where $\alpha, a \in (0, 1)$. Direct calculation of the Allen-Uzawa elasticities yields that $\sigma_{13} = \sigma_{23} = \sigma < 1$, while $\sigma_{12} = 1 + (1 - \sigma) \frac{1-s}{s} > 1$, where s is the expenditure share on the Cobb-Douglas bundle. Hence, this sector is not inelastic. However, it is simple to show that the sector's log-cost function is globally convex, and hence that the sector is fragile. $\triangle$

6 Application II: Shock Propagation

Understanding the effects of small shocks is a classic analytical method in macroeconomic analysis. This is for the reason that, by focusing on small shocks, non-linearities can be ignored. We will find, however, that the presence of technology switching implies that non-linearities affect even such a local analysis, caveating the standard takeaway from [Hulten \(1978\)](#): observed Domar weights are not generally sufficient to determine the effects of even small shocks without knowledge of the underlying technologies.

6.1 Shock Propagation in the Small

We denote the starting point from which the economy is shocked by $x \in \mathbb{R}^n$, the direction of a productivity shock by $d \in \mathbb{R}^n$, and the size of the shock by $t \in \mathbb{R}_{++}$. We define the local effect of a shock in direction d of size t starting from x as:

$$\Pi_0(x; d) = \lim_{t \rightarrow 0^+} \frac{\Pi(x + td) - \Pi(x)}{t}. \quad (53)$$

We say that Π is *directionally differentiable* at x if this limit exists for every d and that Π is *Gateaux differentiable* at x if this limit is a linear function of d .¹⁴

We define the optimal technology choice profile across sectors from the endogenous Leontief representation (see [Proposition 1](#)) as:

$$\Sigma(p) = \left\{ \tilde{W} \in \mathcal{W} : \tilde{w}_i \in \arg \min_{\hat{w}_i \in \Delta} \frac{\sum_{j=1}^n \hat{w}_{ij} \exp(p_j)}{G_i(\hat{w}_i)}, \forall i \in [n] \right\} \quad \forall p \in \mathbb{R}^n \quad (54)$$

Intuitively, $\Sigma(p)$ corresponds to the set of cost-minimizing technology choices across sectors when they face the log-price vector p . We also define the operator $L : \mathcal{W} \times \mathbb{R}^n \rightarrow \mathcal{W}$:

$$L_{ij}(\tilde{W}, p) = \frac{\tilde{w}_{ij} \exp(p_j)}{\sum_{k=1}^n \tilde{w}_{ik} \exp(p_k)} \quad \forall i, j \in [n] \quad (55)$$

which returns the Cobb-Douglas expenditure shares given a technology choice matrix $\tilde{W}$. We

¹⁴All these properties and derived objects are analogously defined for the log-cost operator T and the real GDP function Υ .

can now derive the formula for the directional derivative of equilibrium log-prices:

Theorem 2. *The equilibrium log-price operator Π is directionally differentiable with:*

$$\Pi_0(x; d) = \min_{\tilde{W} \in \Sigma(\Pi(x))} \Omega_{L(\tilde{W}, \Pi(x))} d \quad \forall x, d \in \mathbb{R}^n \quad (56)$$

This formula says that for every direction d , shocks propagate according to the Leontief inverse of a fictitious Cobb-Douglas economy with expenditure shares given by L . Moreover, among all of the cost-minimizing technologies that prevail at the equilibrium log-prices $\Pi(x)$, the fictitious Cobb-Douglas economy is the one that minimizes equilibrium prices following the shock.

We emphasize that different directions of shocks can give rise to different technology choices of sectors, causing the derivative of prices to jump. This happens at exactly those points where multiple technologies are optimal for some sector. The following four equivalent statements characterize the points at which technology switching introduces jumps.

Proposition 6 (When Jumps Feature). *Fix $x \in \mathbb{R}^n$. The following statements are equivalent:*

- (i) *T is Gateaux differentiable at $\Pi(x)$*
- (ii) *Π is Gateaux differentiable at x*
- (iii) *$\Sigma(\Pi(x))$ is a singleton*
- (iv) *There exists a unique competitive equilibrium at x .*

Moreover, for almost all $x \in \mathbb{R}^n$, these four equivalent conditions are true.

Thus, whenever there are multiple competitive equilibria or T is not Gateaux differentiable at equilibrium, jumps occur and local approximations may serve as a poor guide. While these jumps arise in general, if all sectors of the economy are fragile, then T is Gateaux differentiable and hence such economies always feature a unique equilibrium. Away from the fragile case in which jumps are not possible, this result clarifies that jump points comprise a zero-measure subset of $\mathbb{R}^n$. Despite this seeming rarity, their presence can still lead to arbitrarily large errors if the analyst ignores them. As we will see in our next application, jumps are essential to computing the effects of shock volatility on expected GDP.

6.2 The Insufficiency of Local Domar Weights

We now use Theorem 2 to derive the directional derivative of equilibrium output $\Upsilon_0(x; d) = \lim_{t \downarrow 0} \frac{\Upsilon(x+td) - \Upsilon(x)}{t}$. We provide a generalized Hulten's theorem: the change in aggregate labor productivity induced by infinitesimal productivity shocks is given by the sales share vector (or Domar weights). To this end, let $\mathcal{E}(x)$ be the set of competitive equilibria at $x \in \mathbb{R}^n$. For any competitive equilibrium $E \in \mathcal{E}(x)$, we define the Domar weight vector as $\lambda_E = (P_i^E Y_i^E)_{i=1}^n$.

Proposition 7 (Generalized Hulten’s Theorem). *Equilibrium log output Υ is directionally differentiable with derivative:*

$$\Upsilon_0(x; d) = \max_{E \in \mathcal{E}(x)} \lambda'_E d_Z \quad \forall x, d \in \mathbb{R}^n \quad (57)$$

where $d_Z = -(1 - \beta)d$ is the direction of the shock expressed in units of log-TFP.

This generalizes Hulten’s theorem beyond the case in which sectors’ cost functions are differentiable. The technology switching that previously led to discontinuity in the response of prices to shocks also leads to jumps in Domar weights. Intuitively, when the economy is at a technology switching point, different directions of shocks will propagate differently.

Proposition 7 implies that knowledge of $\mathcal{E}(x)$ is essential to know the effects of local productivity shocks. Thus, except when there are unique equilibrium Domar weights, knowledge of Domar weights before a shock is insufficient to know the effects of a shock. The analyst must know how technology may change in order to predict how shocks propagate. For large shocks, it is well understood that [Hulten \(1978\)](#) may provide a poor approximation ([Baqaee and Farhi, 2019](#)). What is new is that Proposition 7 shows that the same conclusion can be true even for small shocks. Moreover, Proposition 6 clarifies exactly when the conclusion of [Hulten \(1978\)](#) fails: when technology switching is fast, technologies used before a shock can be a poor guide to technologies used following a shock. Conversely, when the economy is fragile (and thus by Proposition 3 has a Gateaux differentiable T), the conclusion of [Hulten \(1978\)](#) is valid.

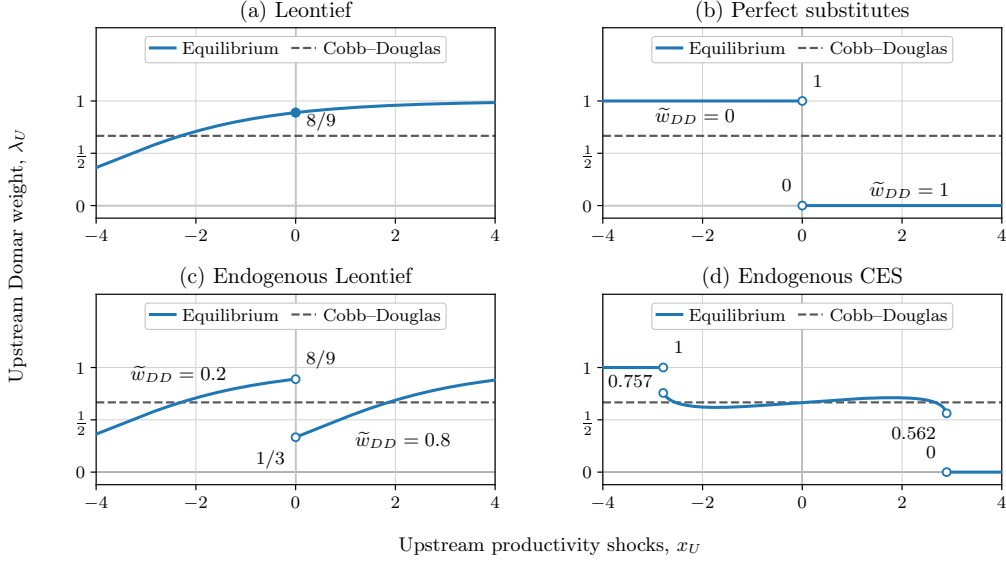
6.3 Illustrating the Generalized Hulten’s Theorem

To understand Proposition 7, we apply the formula in the example economies that we used to illustrate Theorem 1. Consider the local effects of a shock to upstream log-productivity x_U starting from a steady state in which both the upstream and downstream sectors are equally productive $(Z_U, Z_D) = (1, 1)$. At the initial productivity vector, by normalization, the prices of both sectors are equal. Hence, the expenditure share operator simply reduces to $L_{ij}(\tilde{W}, \Pi(0)) = \tilde{w}_{ij}$. Therefore, the Domar weight vector given technology choices $\tilde{W}$ is given by $\lambda' = \frac{1}{1-\beta} \xi' \Omega_{L(\tilde{W}, \Pi(0))} = \frac{1}{1-\beta} \xi' \Omega_{\tilde{W}}$. As $\xi_D = 1$ (the household only consumes downstream goods) and $\tilde{w}_{UU} = 1$ (the upstream sector has no technology choice in this example), we have that the Domar weight vector corresponding to downstream technology choice $\tilde{w}_{DD}$ is given by:

$$\lambda'(\tilde{w}_{DD}) = 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}' \begin{pmatrix} 1 & 0 \\ \frac{1-\tilde{w}_{DD}}{2-\tilde{w}_{DD}} & \frac{1}{2-\tilde{w}_{DD}} \end{pmatrix} = 2 \begin{pmatrix} \frac{1-\tilde{w}_{DD}}{2-\tilde{w}_{DD}} & \frac{1}{2-\tilde{w}_{DD}} \end{pmatrix} \quad (58)$$

As efficient technology choice for the downstream sector at symmetry simply maximizes downstream output, we have that $\tilde{w}_{DD}$ lies in the set $S_D = \{\tilde{w}_{DD} \in [0, 1] : G_D(\tilde{w}_{DD}, 1 - \tilde{w}_{DD}) = 1\}$.

Figure 7: Equilibrium Domar Weights in Various Economies



Notes: Each panel plots the equilibrium Domar weights of the upstream sector in various economies as the blue line. The dashed gray line is the (constant) equilibrium upstream Domar weight in a Cobb-Douglas economy in which downstream has equal expenditure shares on upstream and downstream inputs. Throughout, $\beta = 0.5$ and $x_D = 0$. In panel (a), technology is Leontief with a downstream input share of $\omega = 0.2$. In panel (b), technology is perfect substitutes between upstream and downstream. In panel (c), downstream can choose between Leontief technologies with downstream input shares $\omega \in \{0.2, 0.8\}$. In panel (d), downstream technology is endogenous CES with $\alpha_0 = 0.5$, $\sigma_0 = 0.25$, $\kappa_\alpha = 8$, and $\kappa_\sigma = 16$, $\alpha \in [0, 1]$, $\sigma \in [0, 1]$.

We denote the largest and smallest elements of this set by $\bar{w}_{DD}$ and $\underline{w}_{DD}$. Proposition 7 implies:

$$\Upsilon_0(0; (x_U, 0)) = \max_{\tilde{w}_{DD} \in S_D} \left\{ \frac{1 - \tilde{w}_{DD}}{2 - \tilde{w}_{DD}} (-x_U) \right\} = \begin{cases} \frac{1 - \underline{w}_{DD}}{2 - \underline{w}_{DD}} (-x_U), & x_U \leq 0, \\ \frac{1 - \bar{w}_{DD}}{2 - \bar{w}_{DD}} (-x_U), & x_U > 0. \end{cases} \quad (59)$$

Intuitively, when there is an adverse shock to the upstream sector (*i.e.*, $x_U > 0$), the downstream sector switches to the technology (among all of those that are efficient) that is least reliant on upstream inputs. Conversely, when there is a positive shock to the upstream sector, it switches technology to be maximally exposed to the shock. Thus, there is a discontinuity in the effects of shocks when there are multiple efficient technology choices.

We illustrate this (in)sufficiency of local Domar weights through a series of examples.

Example 5 (continuing from p. 19). If there is a single Leontief technology downstream with downstream input weight α , the unique efficient technology is $\tilde{w}_{DD} = \alpha$. Hence, $S_D = \{\alpha\}$ and there is no discontinuity. This is illustrated in panel (a) of Figure 7: upstream's Domar weight smoothly and monotonically increases as it becomes less productive. $\triangle$

Example 6 (continuing from p. 19). If there is a perfect substitutes technology downstream, then all technologies are efficient $S_D = [0, 1]$, and there are highly asymmetric effects: positive shocks propagate while negative shocks generate no adverse consequences. This is illustrated in panel (b) of Figure 7: adverse shocks do not propagate as the Domar weight of upstream is zero while beneficial shocks propagate one-for-one as the Domar weight of upstream is one. $\triangle$

Example 7 (continuing from p. 20). More generally, if there are finitely many Leontief technologies downstream, negative shocks propagate according to the smallest Leontief weight on upstream and positive shocks propagate according to the largest Leontief weight on upstream. We illustrate this in our two technology running example in panel (c) of Figure 7. Starting from a situation in which upstream is more productive than downstream ($x_U < 0$), worsened productivity in upstream (higher x_U) increases the Domar weight of upstream. However, at the point at which downstream switches from using upstream more intensively to less intensively ($x_U = 0$), the Domar weight of upstream jumps down, before once again increasing. $\triangle$

Example 4 (continuing from p. 12). Panel (d) of Figure 7 shows equilibrium Domar weights in this endogenous CES example. When upstream is much more productive than downstream, downstream uses only upstream inputs ($\alpha^* = 0$) and the upstream Domar weight is one. At a critical point, even though each CES technology is smooth and the opportunity cost of changing technologies is smooth, the optimal technology choice jumps to using some strictly positive amount of downstream goods ($\alpha^* > 0$), leading to a downward jump in the Domar weights. After this point, technological change in both input intensity α and the substitution elasticity σ occurs smoothly and the upstream Domar weight first declines (as effective substitution dominates), before rising (as effective complementarity dominates), and then declines again. Finally, at a critical point where upstream is sufficiently unproductive, downstream ceases to use upstream inputs at all ($\alpha^* = 1$) and the upstream Domar weight jumps to zero. $\triangle$

These examples illustrate that resilience and the ability to switch technologies can imply rapid changes in Domar weights that are not captured by local information about the economy.

7 Application III: Microeconomic Volatility and GDP

We finally apply our results to study how changes in microeconomic volatility affect the average GDP of the economy.

7.1 Set-up and Defining Local Resilience and Fragility

For this section, we assume that sectoral log productivities are stochastic and study how a change in the variance of the log productivity of an arbitrary sector $i \in [n]$ affects average

log GDP. To this end, suppose that sector i 's (normalized and inverse) log productivity can be written as $X_i = \mu_i + s_i \nu_i$, where $\mu_i \in \mathbb{R}$ represents mean log-productivity, $s_i \in \mathbb{R}_{++}$ denotes the standard deviation of log productivity and ν_i is a shock (in standardized units) to log productivity. We assume that ν_i is (jointly) independent across sectors and has probability density function φ_i , which is even, continuous, and has unit variance. With an abuse of notation, we henceforth suppress dependence on means, and let $\tilde{\varphi}_s(x)$ and, for all $i \in [n]$, $\tilde{\varphi}_{-i,s}(x_{-i})$ denote the induced densities of the random vectors $X = (X_j)_{j \in [n]}$ and $X_{-i} = (X_j)_{j \neq i}$. A key example accommodated by this setting is that X is jointly normally distributed with mean μ and variance-covariance matrix $\text{diag}(s^2)$, where $s^2 = (s_1^2, \dots, s_n^2) \in \mathbb{R}_{++}^n$.

We define expected log GDP given sectoral standard deviations $s \in \mathbb{R}_{++}^n$ as:

$$Y(s) = \mathbb{E}_s[\Upsilon(X)] = \int_{\mathbb{R}^n} \Upsilon(x) \tilde{\varphi}_s(x) dx = \int_{\mathbb{R}^n} \Upsilon(x) \left(\prod_{i=1}^n \frac{1}{s_i} \varphi_i\left(\frac{x_i - \mu_i}{s_i}\right) \right) dx \quad (60)$$

We say that the economy is locally (at s) fragile to increases in the volatility from sector i if $\frac{\partial Y(s)}{\partial s_i} \leq 0$ and the economy is locally resilient if $\frac{\partial Y(s)}{\partial s_i} \geq 0$. It also follows that if the economy is resilient (fragile), then the economy is also everywhere locally resilient (fragile) to every sector i .¹⁵ We can thus consider $\frac{\partial Y(s)}{\partial s_i}$ as a quantitative measure of the local resilience of the economy to sector i . We now proceed to provide a general decomposition of this quantitative measure.

7.2 The Effects of Microeconomic Volatility on GDP

To state the result, we define $\mathcal{D}_T$ as the set of $x \in \mathbb{R}^n$ such that T is Gateaux differentiable at x . We also define $\mathcal{H}_T$ as the set of $x \in \mathbb{R}^n$ where T is twice continuously differentiable. In this section, we maintain the following technical assumptions: (i) $\mathcal{D}_T$ is open and (ii) $\mathcal{D}_T = \mathcal{H}_T$. Define the corresponding set of productivity vectors by $\mathcal{D}_\Pi = \Pi^{-1}(\mathcal{D}_T)$. By Proposition 6, $\mathcal{D}_\Pi$ is the set of productivity vectors admitting a unique equilibrium. Its complement has Lebesgue measure zero, so it has probability one under every shock distribution considered here. Under the lens of Proposition 6, the first part of the assumption says that equilibrium uniqueness is robust to small perturbations of shocks. The second part of the assumption states that whenever a unique equilibrium exists, the log-cost operator admits a second-order expansion. Importantly, this does not imply that the second-order expansion approximates well the equilibrium *everywhere*, just generically. While this might seem a detail, especially since we are considering absolutely continuous shock distributions, our decomposition below shows that the (non-generic) discontinuous changes in the derivative of T still matter. Intuitively, while jumps are rare, they are locally infinitely larger than the smooth variation in the function.

¹⁵For every convex function $\phi : \mathbb{R} \rightarrow \mathbb{R}$, the map $s_i \mapsto \int_{\mathbb{R}} \phi(\mu_i + s_i \nu_i) \varphi_i(\nu_i) d\nu_i$ is non-decreasing over $\mathbb{R}_+$, for every $i \in [n]$. Thus, the distribution of each X_i increases in the convex order whenever s_i increases.

For each sector $i \in [n]$, we define the coordinatewise minimum Domar weight:

$$\lambda_i(x) = \min_{E \in \mathcal{E}(x)} \lambda_{E,i}, \quad x \in \mathbb{R}^n, \quad (61)$$

On $\mathcal{D}_\Pi$, this coincides with the Domar weight of the unique equilibrium. Under our maintained assumptions, for every $x \in \mathcal{D}_\Pi$, their derivatives $\left(\frac{\partial \lambda_i(x)}{\partial x_i}\right)_{i \in [n]} \in \mathbb{R}^n$ are well defined. We set $\frac{\partial \lambda_i(x)}{\partial x_i} = 0$ for all $x \notin \mathcal{D}_\Pi$ and all $i \in [n]$. Moreover, for every sector $i \in [n]$, define the Domar weight jump as $\Delta \lambda_i(x) = \lambda_i(x) - \lim_{k \rightarrow \infty} \lambda_i(x - \frac{1}{k} e^i)$ for all $x \in \mathbb{R}^n$. These jumps are well defined everywhere, and they are equal to 0 on $\mathcal{D}_\Pi$.¹⁶ Finally, we define the function $m_i : \mathbb{R} \rightarrow \mathbb{R}_+$:

$$m_i(x_i) = \int_{\left|\frac{x_i - \mu_i}{s_i}\right|}^{\infty} r \varphi_i(r) dr \quad (62)$$

Intuitively, this object corresponds to the expected value of sector i 's productivity multiplied by an indicator that productivity is x_i standard deviations above the mean. For a Gaussian random variable, $X_i \sim N(\mu_i, s_i^2)$, this object satisfies $m_i(x_i) = \varphi\left(\frac{x_i - \mu_i}{s_i}\right)$ where φ is the standard normal density, and is just the probability density of productivity shocks scaled by their dispersion. With these objects in hand, we can characterize the local resilience of the economy as follows:

Theorem 3. *For all $s > 0$ and $i \in [n]$, we have.*¹⁷

$$\frac{\partial Y(s)}{\partial s_i} = (1 - \beta) \mathbb{E}_s \left[- \int_{\mathbb{R}} \frac{\partial \lambda_i(x_i, X_{-i})}{\partial x_i} m_i(x_i) dx_i - \sum_{x_i \in \mathbb{R}} \Delta \lambda_i(x_i, X_{-i}) m_i(x_i) \right] \quad (63)$$

Moreover, we have that $-\Delta \lambda_i(x) \geq 0$ for all $x \in \mathbb{R}^n$.

We first show that the effect of changing the variance of sector i on average GDP is simply the average of the second-order distributional derivative of GDP. We then obtain this result by showing that this second-order distributional derivative can be decomposed into two terms. The first term is the standard and familiar second-order term from [Baqee and Farhi \(2019\)](#): the second-order effects of changing productivity shocks are captured by the change in the Domar weights. However, in general, this does not capture the total effect. The second term weighs points at which the Domar weights jump because sectors switch technologies discontinuously. This term always contributes positively. Intuitively, technology switching always makes the

¹⁶To see why $\Delta \lambda_i(x)$ is well defined, observe that by Proposition 10, the function $g(t) = \Upsilon(x + te^i) + \frac{\beta}{2(1-\beta)} t^2$ is convex. By Proposition 7, $\lambda_i(x + te^i) = -\frac{g'_+(t) - \frac{\beta}{1-\beta} t}{1-\beta}$ for all $t \in \mathbb{R}$. The right derivative g'_+ is finite and nondecreasing. Hence the sequence $g'_+(-1/k)$ is nondecreasing and bounded above by $g'_+(0)$, so it has a finite limit. For a convex function, this limit equals $g'_-(0)$. It follows that $\lim_{k \rightarrow \infty} \lambda_i(x - \frac{1}{k} e^i) = -\frac{g'_-(0)}{1-\beta}$.

¹⁷The expectation is over X_{-i} , and the sum is over the at most countably many points where $\Delta \lambda_i(x_i, X_{-i}) \neq 0$.

economy more locally resilient because sectors change technologies efficiently. Thus, neglecting technology switching will always *overstate* the extent of the economy’s fragility.

We now unpack these terms and how they contribute to local resilience and fragility through some special cases and examples.

Remark 4 (Expected Changes in Domar Weights are Sufficient with Smooth Technology). Suppose that each sector has a sufficiently “smooth” production technology in the precise sense that its log-cost is Gateaux differentiable and that $X_i \sim N(\mu_i, s_i^2)$. In this case, there is always a uniquely optimal technological choice and the jump component in Theorem 3 is identically zero. That is, we have (using the fact that $m_i(x_i) = \varphi\left(\frac{x_i - \mu_i}{s_i}\right)$ for Gaussian shocks):

$$\frac{\partial Y(s)}{\partial s_i} = -(1 - \beta)s_i \mathbb{E}_s \left[\frac{\partial \lambda_i(X)}{\partial x_i} \right] \quad (64)$$

A direct corollary of this fact is that if all sectors are fragile, only the first term appears and the total effect of microeconomic volatility is negative.¹⁸ Under this smooth technology, the Baqaee and Farhi (2019) logic of studying the derivatives of Domar weights is sufficient to characterize local resilience and fragility. However, even in this case, we highlight that the formula provided by Theorem 3 is novel and extends the logic from Baqaee and Farhi (2019): it provides the exact—rather than approximate—effects of changing volatility on expected GDP. $\triangle$

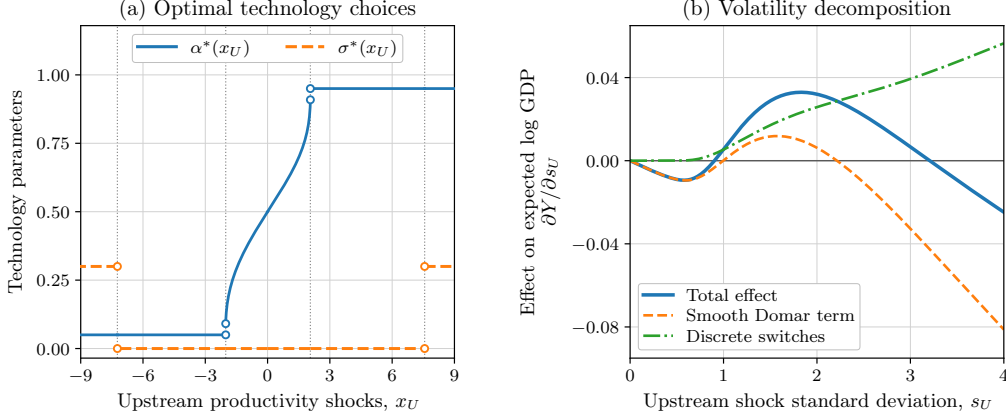
We now explore the more general case with contributions from both smooth changes in Domar weights and technology switching using our running endogenous CES example economy:

Example 4 (continuing from p. 12). In panel (b) of Figure 8, we plot the effect of changing upstream productivity volatility on average log-GDP (solid blue line) in an endogenous CES economy. We recover an S-shaped response of the economy (as in the example from Section 2): volatility is initially harmful, then helpful, and then harmful again. Underlying this total effect are changes in effective substitutability of inputs that arise from smooth changes in optimal input weights α^* and the elasticity of substitution σ^* and discrete changes arising from discontinuous adjustment of technology. We illustrate these technological adjustments in panel (a) of Figure 8. Intuitively, as shocks become larger, downstream both adapts to place less weight on the relatively more expensive sector and becomes more flexible.

Consider first the contribution from smooth changes in Domar weights (orange line). As inputs for fixed technology are gross complements (as $\sigma^* < 1$), if technological changes are insufficiently fast, then the economy is locally fragile and volatility has negative effects. Conversely, even though inputs are always gross complements for any fixed technology, sufficiently fast changes in technology can generate enough substitution to generate local resilience.

¹⁸This is true more generally since in fragile economies Υ is a concave differentiable function (see Corollary 2 and Footnote 11). By contrast, in resilient economies, the derivative is always nonnegative.

Figure 8: The Effects of Microeconomic Volatility in an Endogenous CES Economy



Notes: Panel (a) plots equilibrium technology choices of the downstream sector with endogenous CES technology as a function of upstream productivity shocks. Open circles denote discontinuities in the optimal technology choice. Panel (b) plots the effect of upstream productivity volatility on expected log GDP and decomposes this into the two terms from Theorem 3. Parameters: $-\ln A(\alpha, \sigma) = 2.8(\alpha - 0.5)^2 + 0.05[(\sigma + 24.5)^2 - 24.5^2]$, $\alpha \in [0.05, 0.95]$, $\sigma \in [0, 0.3]$, $X_U \sim N(0, s_U^2)$, $X_D = 0$.

We now turn to how jumps in Domar weights contribute to local resilience (green line). We observe that there are four critical values of shocks to upstream productivity at which downstream finds it optimal for its technology to jump. While the probability of being at these jumps is zero, because they are large, their contribution to the effect of variance on expected log-GDP is non-negligible. Moreover, as guaranteed by Theorem 3, this contribution is always positive. Intuitively, and in line with the less formal logic of our leading example (see Section 2), as the economy will always operate using the most efficient technology, discrete changes in technology create a form of option value. $\triangle$

As this example demonstrates, Theorem 3 formalizes and generalizes the qualitative phenomenon illustrated in Section 2. That is, the effect of microeconomic volatility on expected log-GDP is generally non-monotone. In smooth economies, these effects are summarized by expected changes in Domar weights. In discrete economies, such as an endogenous Cobb-Douglas economy with finitely many technologies, these effects are governed only by jumps in Domar weights. In general, both smooth and discontinuous changes in Domar weights contribute.

8 Conclusion

In this paper, we have provided a global analysis of the canonical production network model with arbitrary CRS technology. Our main result provides a primitive formula for equilibrium

prices that applies in any such economy. As a formula was previously known only in the Cobb-Douglas case (Acemoglu et al., 2012), this formula represents a substantial generalization and applies to popular special cases, such as nested CES economies (Baqee and Farhi, 2019; Dew-Becker, 2023), endogenous Cobb-Douglas economies (Acemoglu and Azar, 2020; Kopytov et al., 2024), and many others (including those studied by Chodorow-Reich et al., 2026; Acemoglu and Tahbaz-Salehi, 2025).

Beyond the technical contribution of this result, we build upon it in three applications. First, we studied the question of when the economy is resilient and when it is fragile. We found that the “min” and the “max” in the general representation correspond in a natural way to the competing forces of resilience and fragility. We also characterized the microeconomic properties of resilience and fragility at the sector level, revealing that standard specifications and approximation-based methods allow only for global resilience or fragility. In the process, we axiomatize endogenous Cobb-Douglas and a form of task-based production technology as equivalent to the properties of resilience and fragility, respectively. Thus, such models cannot capture the qualitative possibility that changing input mix can lead to volatility having “S-shaped” effects on the economy, whereby small and large levels of productivity dispersion are bad, but intermediate levels are good.

Second, we provided a generalization of Hulten’s (1978) theorem to cover the case of general (*i.e.*, non-differentiable) production technology. As we showed, such non-differentiabilities are a natural consequence of allowing for endogenous production networks. Moreover, the generalized theorem places important economic nuance on the received wisdom following Hulten (1978) that, for small shocks, it is not important to know anything about technology or the input-output network. Indeed, the presence of technology switching—which cannot be inferred from mere observation of the current Domar weights—leads current Domar weights to provide an inaccurate description of shock propagation, even for small shocks.

Third, we characterized the effects of microeconomic volatility on GDP. We showed that these effects can always be decomposed into two terms, reflecting how changes in productivity smoothly change the economy’s Domar weights—providing an exact version of the logic from Baqee and Farhi (2019)—and an additional term reflecting how jumps in Domar weights contribute. This change from jumps in Domar weights results from technology switching and always contributes positively. Hence, by this notion of local resilience, economies are necessarily weakly more resilient than would be suggested by local changes in Domar weights (and strictly more so exactly when technology switching occurs in equilibrium).

Finally, we remark that while we have considered a model with a single primary factor and CRS, our analysis nonetheless has implications for models with multiple primary factors and/or decreasing returns to scale (which can always be recast as CRS economies with multiple primary factors). In particular, our price characterization still applies *given* the price vector

for the primary inputs. Thus, applying our methods and results to multi-factor economies represents a promising avenue for future research.

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Appendices

Preliminaries. In all appendices, with an abuse of notation, we often let $x \in \mathbb{R}^n$ denote an arbitrary n -dimensional vector, which does not need to be related to the vector of log-productivity shocks considered in the main text. We endow $\mathbb{R}^n$ with the supnorm $\|x\|_\infty = \max_{i \in [n]} |x_i|$. We say that an operator $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is nonexpansive if $\|S(x) - S(x')\|_\infty \leq \|x - x'\|_\infty$. Given a nonexpansive operator $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\beta \in (0, 1)$, we introduce the new operator $\tilde{S}_\beta : \mathbb{R}^n \rightarrow \mathbb{R}^n$ that maps each $x \in \mathbb{R}^n$ to the solution of the following equation (in $\tilde{x}$):

$$\tilde{x} = (1 - \beta)x + \beta S(\tilde{x}). \quad (65)$$

Lemma 3 in [Cerrei-Vioglio et al. \(2026\)](#) establishes that $\tilde{S}_\beta$ is well defined.

A Proofs of the Results in Section 4

Proof of Proposition 1. (i) $\implies$ (ii) Fix $P \in \mathbb{R}_{++}^n$. Fix also $i \in [n]$ and $\tilde{w}_i \in \Delta$ such that $G_i(\tilde{w}_i) \neq 0$. Define $Q_i = \frac{\tilde{w}_i}{G_i(\tilde{w}_i)} \in \mathbb{R}_+^n$. Since G_i is positive homogeneous, we have that $G_i(Q_i) = 1$. By definition of Q_i , it follows that

$$\frac{\sum_{j=1}^n \tilde{w}_{ij} P_j}{G_i(\tilde{w}_i)} = \sum_{j=1}^n Q_{ij} P_j \geq K_i(P). \quad (66)$$

Because $\tilde{w}_i$ was arbitrarily chosen, it follows that

$$K_i(P) \leq \min_{\tilde{w}_i \in \Delta: G_i(\tilde{w}_i) \neq 0} \frac{\sum_{j=1}^n \tilde{w}_{ij} P_j}{G_i(\tilde{w}_i)} = \min_{\tilde{w}_i \in \Delta} \frac{\sum_{j=1}^n \tilde{w}_{ij} P_j}{G_i(\tilde{w}_i)}. \quad (67)$$

Next, fix Q_i such that $G_i(Q_i) \geq 1$ and define $w_i = \frac{1}{\sum_{j=1}^n Q_{ij}} Q_i$. It follows that

$$\sum_{j=1}^n Q_{ij} P_j \geq \frac{\sum_{j=1}^n Q_{ij} P_j}{G_i(Q_i)} = \frac{\sum_{j=1}^n w_{ij} P_j}{G_i(w_i)} \geq \min_{\tilde{w}_i \in \Delta} \frac{\sum_{j=1}^n \tilde{w}_{ij} P_j}{G_i(\tilde{w}_i)} \quad (68)$$

where the equality follows from positive homogeneity of G_i . Because Q_i was arbitrarily chosen, it follows that $K_i(P) \geq \min_{\tilde{w}_i \in \Delta} \frac{\sum_{j=1}^n \tilde{w}_{ij} P_j}{G_i(\tilde{w}_i)}$, proving $K_i(P) = \min_{\tilde{w}_i \in \Delta} \frac{\sum_{j=1}^n \tilde{w}_{ij} P_j}{G_i(\tilde{w}_i)}$.

By the definition of T , we have

$$T_i(p) = \min_{\tilde{w}_i \in \Delta} \left[\ln \left(\sum_{j=1}^n \tilde{w}_{ij} \exp(p_j) \right) + \ln \left(\frac{1}{G_i(\tilde{w}_i)} \right) \right]. \quad (69)$$

By [Maccheroni et al. \(2006\)](#), this establishes that T is normalized, monotone, and translation invariant, completing the proof of this part.

(ii) $\implies$ (i) follows directly by Theorem 4.

The exact expression for T follows by combining the above with Proposition 1.4.2 in [Dupuis](#)

and Ellis (2011).¹⁹

□

Define the function $H : [0, \infty) \times \Delta \times \Delta \rightarrow [0, \infty]$ as

$$H(\lambda, w_i, \tilde{w}_i) = \begin{cases} 0 & \text{if } w_i = \tilde{w}_i \text{ and } \lambda = 0, \\ \lambda \sum_{j=1}^n w_{ij} \ln \left(\frac{w_{ij}}{\tilde{w}_{ij}} \right) & \text{if } w_i \ll \tilde{w}_i \text{ and } \lambda \in (0, \infty) \\ \infty & \text{otherwise.} \end{cases} \quad (70)$$

Lemma 2. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be such that there exists $\lambda \in [0, \infty)$ and, for every $i \in [n]$, a lower semicontinuous function $c_i : \Delta \rightarrow [0, \infty]$ with

$$T_i(x') = \min_{\tilde{w}_i \in \Delta} \max_{w_i \in \Delta} \{ \langle w_i, x' \rangle - H(\lambda, w_i, \tilde{w}_i) + c_i(\tilde{w}_i) \} \quad \forall x' \in \mathbb{R}^n, \quad (71)$$

with the convention that if $c_i(\tilde{w}_i) = \infty$, then $-H(\lambda, w_i, \tilde{w}_i) + c_i(\tilde{w}_i) = \infty$. We have that

$$\Pi(x) = \min_{\tilde{W} \in \mathcal{W}} \max_{W \in \mathcal{W}} \left\{ \Omega_W \left(x - \tau^c(W \| \tilde{W}) \right) \right\} \quad \forall x \in \mathbb{R}^n \quad (72)$$

where $\tau_i^c(W \| \tilde{W}) = \frac{\beta}{1-\beta} (H(\lambda, w_i, \tilde{w}_i) - c_i(\tilde{w}_i))$ for all $i \in [n]$, with the convention that for all $\tilde{W} \in \mathcal{W}$, if there is $i \in [n]$ such that $c_i(\tilde{w}_i) = \infty$, then $\tau_j^c(W \| \tilde{W}) = -\infty$ for all $j \in [n]$ and all $W \in \mathcal{W}$.

Proof. For every $i \in [n]$, fix $c_i : \Delta \rightarrow [0, \infty]$ as in the statement. Consider first $\lambda \in (0, \infty)$. For all $\tilde{W}, W \in \mathcal{W}$ such that, for all $i \in [n]$, $c_i(\tilde{w}_i) < \infty$, and $w_i \ll \tilde{w}_i$, define $\alpha(W, \tilde{W}) \in \mathbb{R}^n$ as

$$\alpha(W, \tilde{W})_i = -H(\lambda, w_i, \tilde{w}_i) + c_i(\tilde{w}_i) \quad \forall i \in [n] \quad (73)$$

and the operator $T_{\tilde{W}, W} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ as

$$T_{\tilde{W}, W}(x') = Wx' + \alpha(W, \tilde{W}). \quad (74)$$

Also, for every $\tilde{W} \in \mathcal{W}$ such that, for all $i \in [n]$, $c_i(\tilde{w}_i) < \infty$, define the operator $T_{\tilde{W}}^* : \mathbb{R}^n \rightarrow \mathbb{R}^n$ as

$$T_{\tilde{W}}^*(x') = \max_{W : \forall i \in [n], w_i \ll \tilde{w}_i} T_{\tilde{W}, W}(x'). \quad (75)$$

Clearly, each $T_{\tilde{W}}^*$ so defined is non-decreasing and translation invariant, hence nonexpansive. Next, observe that

$$T(x') = \min_{\tilde{W} : \forall i \in [n], c_i(\tilde{w}_i) < \infty} T_{\tilde{W}}^*(x') \quad \forall x' \in \mathbb{R}^n. \quad (76)$$

By Remark 1 in Cerreia-Vioglio et al. (2026), for all $\tilde{W}, W \in \mathcal{W}$, such that, for all $i \in [n]$,

¹⁹The steps of the proof of that result need to be adapted in the obvious way given that we have $\ln \left(\sum_{j=1}^n \tilde{w}_{ij} \exp(p_j) \right)$ in place of $-\ln \left(\sum_{j=1}^n \tilde{w}_{ij} \exp(-p_j) \right)$.

$c_i(\tilde{w}_i) < \infty$, and $w_i \ll \tilde{w}_i$, we have

$$\tilde{T}_{\tilde{W}, W, \beta}(x) = \Omega_W \left(x + \frac{\beta}{1-\beta} \alpha(W, \tilde{W}) \right) \quad \forall x \in \mathbb{R}^n. \quad (77)$$

Next, fix $\tilde{W} \in \mathcal{W}$ such that, for all $i \in [n]$, $c_i(\tilde{w}_i) < \infty$ and observe that, for all $W \in \mathcal{W}$ such that $w_i \ll \tilde{w}_i$ and all $x' \in \mathbb{R}^n$, we have $T_{\tilde{W}, W}(x') \leq T_{\tilde{W}}^*(x')$. By Lemma 5 in [Cerrea-Vioglio et al. \(2026\)](#), we also have that

$$\Omega_W \left(x + \frac{\beta}{1-\beta} \alpha(W, \tilde{W}) \right) = \tilde{T}_{\tilde{W}, W, \beta}(x) \leq \tilde{T}_{\tilde{W}, \beta}^*(x) \quad (78)$$

for all $W \in \mathcal{W}$ such that $w_i \ll \tilde{w}_i$ for all $i \in [n]$ and $x \in \mathbb{R}^n$. By Equation 75, for all $x \in \mathbb{R}^n$, there exists $\hat{W} \in \mathcal{W}$ such that, for all $i \in [n]$, $c_i(\tilde{w}_i) < \infty$, and

$$\tilde{T}_{\tilde{W}, \beta}^*(x) = (1-\beta)x + \beta T_{\tilde{W}}^*(\tilde{T}_{\tilde{W}, \beta}^*(x)) = (1-\beta)x + \beta T_{\tilde{W}, \hat{W}}(\tilde{T}_{\tilde{W}, \beta}^*(x)) \quad (79)$$

yielding that $\tilde{T}_{\tilde{W}, \beta}^*(x) = \tilde{T}_{\tilde{W}, \hat{W}, \beta}(x)$, hence that

$$\tilde{T}_{\tilde{W}, \beta}^*(x) = \max_{W \in \mathcal{W}: \forall i \in [n], w_i \ll \tilde{w}_i} \left\{ \Omega_W \left(x + \frac{\beta}{1-\beta} \alpha(W, \tilde{W}) \right) \right\}. \quad (80)$$

Because $\tilde{W}$ was arbitrarily chosen, it follows that Equation 80 holds for all $\tilde{W} \in \mathcal{W}$ such that $c_i(\tilde{w}_i) < \infty$ for all $i \in [n]$.

Next, fix $x \in \mathbb{R}^n$. For all $\tilde{W} \in \mathcal{W}$ with $c_i(\tilde{w}_i) < \infty$ for all $i \in [n]$, Lemma 5 in [Cerrea-Vioglio et al. \(2026\)](#) implies that

$$\Pi(x) \leq \tilde{T}_{\tilde{W}, \beta}^*(x) \quad (81)$$

because $T \leq T_{\tilde{W}}^*$. By (76), there exists $\tilde{W}^* \in \mathcal{W}$, with $c_i(\tilde{w}_i) < \infty$ for all $i \in [n]$, such that

$$\Pi(x) = (1-\beta)x + \beta T(\Pi(x)) = (1-\beta)x + \beta T_{\tilde{W}^*}^*(\Pi(x)) \quad (82)$$

yielding that $\Pi(x) = \tilde{T}_{\tilde{W}^*, \beta}^*(x)$. This in turn yields that

$$\min_{\tilde{W}: \forall i \in [n], c_i(\tilde{w}_i) < \infty} \tilde{T}_{\tilde{W}, \beta}^*(x) \leq \tilde{T}_{\tilde{W}^*, \beta}^*(x) = \Pi(x) \leq \min_{\tilde{W}: \forall i \in [n], c_i(\tilde{w}_i) < \infty} \tilde{T}_{\tilde{W}, \beta}^*(x), \quad (83)$$

hence that

$$\Pi(x) = \min_{\tilde{W}: \forall i \in [n], c_i(\tilde{w}_i) < \infty} \max_{W: \forall i \in [n], w_i \ll \tilde{w}_i} \left\{ \Omega_W \left(x + \frac{\beta}{1-\beta} \alpha(W, \tilde{W}) \right) \right\} \quad (84)$$

$$= \min_{\tilde{W} \in \mathcal{W}} \max_{W \in \mathcal{W}} \left\{ \Omega_W \left(x - \tau^c(W || \tilde{W}) \right) \right\}, \quad (85)$$

as desired. Finally, suppose $\lambda = 0$. Observe that, for all $i \in [n]$, we have

$$T_i(x') = \min_{\tilde{w}_i \in \Delta} \{ \langle \tilde{w}_i, x' \rangle + c_i(\tilde{w}_i) \} \quad \forall x' \in \mathbb{R}^n. \quad (86)$$

With this, a completely symmetric argument leads to

$$\Pi(x) = \min_{\tilde{W} \in \mathcal{W}} \left\{ \Omega_{\tilde{W}}(x - \tau^c(\tilde{W} || \tilde{W})) \right\} \quad (87)$$

as desired. $\square$

Proof of Theorem 1. (i) implies (ii). Let G be a profile of regular production functions and let T be the log-cost operator induced by it. We first prove that if G induces Π as the equilibrium log-price operator, then Π is given by Equation 31. By Proposition 1, observe that for every $i \in [n]$, T_i can be written as in Equation 71 with $\lambda = 1$ and $c_i(\tilde{w}_i) = \ln \left(\frac{1}{G_i(\tilde{w}_i)} \right)$ for all $\tilde{w}_i \in \Delta$, which is clearly lower semicontinuous. Therefore, Equation 31 immediately follows by Lemma 2 and Equation 30. In turn, this implies that Π is translation invariant and monotone. To see that Π is normalized, observe that $p = 0$ is the unique solution of the equation $p = \beta T(p)$ due to normalization of T (see Proposition 1). To see that Π is bijective first observe that, for all $p \in \mathbb{R}^n$, we can define $x = \frac{p - \beta T(p)}{1 - \beta}$ so that $p = \Pi(x)$, thereby showing that Π is onto. Next, assume by contradiction that $\Pi(x) = \Pi(x')$ for some $x \neq x'$ and observe that

$$x = \frac{\Pi(x) - \beta T(\Pi(x))}{1 - \beta} = \frac{\Pi(x') - \beta T(\Pi(x'))}{1 - \beta} = x', \quad (88)$$

yielding a contradiction. This shows that Π is also injective, hence bijective. Finally, observe that $\tilde{K}_\Pi = \exp \circ \left(\frac{I - (1 - \beta)\Pi^{-1}}{\beta} \right) \circ \ln = \exp \circ T \circ \ln = K_T$, which is quasiconcave by Proposition 1.

(ii) implies (i) Define the operator $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ as $T = \frac{I - (1 - \beta)\Pi^{-1}}{\beta}$. This is well-defined since Π is bijective. If we can show that T is translation invariant, normalized, and such that K_T is quasiconcave, then, by Proposition 1 there exists a profile of regular production functions G that induce T_i as the log-cost operator of each sector i . In turn, by definition of T , we have that Π is the equilibrium log-price operator induced by G , yielding the desired result. With this, we next show that T satisfies all the aforementioned properties. First, observe that, by construction, $K_T = \tilde{K}_\Pi$ is quasiconcave. To see that T is normalized, observe that $\Pi^{-1}(0) = 0$ because Π is normalized and bijective. Thus $T(0) = 0$. To see that T is translation invariant, fix any $p \in \mathbb{R}^n$ and $k \in \mathbb{R}$ and define $x := \Pi^{-1}(p)$. By recalling that $\Pi(x + ke) = \Pi(x) + ke = p + ke$, we have $p + ke = (1 - \beta)(x + ke) + \beta T(p + ke)$. By rearranging this expression, we have

$$T(p + ke) = \frac{p + ke - (1 - \beta)(x + ke)}{\beta} = \frac{p - (1 - \beta)\Pi^{-1}(p)}{\beta} + ke = T(p) + ke. \quad (89)$$

This in turn shows that T also satisfies translation invariance, concluding the proof. $\square$

B Proofs of the Results in Section 5

Proof of Proposition 2. Fix $i \in [n]$ and a log-cost operator T_i .

(i) implies (ii). Because sector i is resilient and T_i is a log-cost operator, by Proposition 1 we have that T_i is normalized, non-decreasing, translation invariant, and concave. By Lemma

26 in [Maccheroni et al. \(2006\)](#), it follows that

$$T_i(p) = \min_{w_i \in \Delta} \{ \langle w_i, p \rangle + d_i(w_i) \} \quad \forall p \in \mathbb{R}^n, \quad (90)$$

where $d_i : \Delta \rightarrow [0, \infty]$ is lower semicontinuous, convex, and such that $\min_{w_i \in \Delta} d_i(w_i) = 0$. In particular, if T_i is positive homogeneous $d_i(w_i) \in \{0, \infty\}$ for all $w_i \in \Delta$. With this, we have

$$K_i(P) = \exp(T_i(\ln(P))) = \min_{w_i \in \Delta} \left\{ \frac{\prod_{j=1}^n P_j^{w_{ij}}}{\exp(-d_i(w_i))} \right\} \quad \forall P \in \mathbb{R}_{++}^n. \quad (91)$$

Next, define

$$G_i^*(Q_i) = \max_{w_i \in \Delta} \left\{ \exp(-d_i(w_i)) \prod_{j=1}^n \left(\frac{Q_{ij}}{w_{ij}} \right)^{w_{ij}} \right\} \quad \forall Q_i \in \mathbb{R}_+^n. \quad (92)$$

Observe that G_i^* is non-decreasing and such that

$$\max_{\tilde{w}_i \in \Delta} G_i^*(\tilde{w}_i) = \max_{w_i \in \Delta} \left\{ \exp(-d_i(w_i)) \max_{\tilde{w}_i \in \Delta} \prod_{j=1}^n \left(\frac{\tilde{w}_{ij}}{w_{ij}} \right)^{w_{ij}} \right\} = 1. \quad (93)$$

Moreover, since the objective function in the maximization of Equation 92 is the product of two upper semicontinuous functions, it is upper semicontinuous in (Q_i, w_i) . In turn, Lemma 17.30 in [Aliprantis and Border \(2006\)](#) implies that G_i^* is upper semicontinuous, and Proposition 5.94 in [Moore \(1999\)](#) implies that it is, in fact, continuous.

By letting $C_i = \text{cl}(\{w_i \in \Delta : d_i(w_i) < \infty\})$ and $A_i(w_i) = \exp(-d_i(w_i))$ for all $w_i \in C_i$, we obtain

$$G_i^*(Q_i) = \max_{w_i \in C_i} \left\{ A_i(w_i) \prod_{j=1}^n \left(\frac{Q_{ij}}{w_{ij}} \right)^{w_{ij}} \right\} \quad \forall Q_i \in \mathbb{R}_+^n. \quad (94)$$

Observe that, given the properties of d_i , we have that C_i is nonempty, compact, and convex, and A_i is upper semicontinuous, log-concave, and such that $\max_{w_i \in C_i} A_i(w_i) = 1$. Moreover, by Lemma 19, we have that G_i^* is regular and concave, hence quasiconcave. Next, it is easy to see that, for all $P \in \mathbb{R}_{++}^n$,

$$\min_{Q_i \in \mathbb{R}_+^n : G_i^*(Q_i) \geq 1} \sum_{j=1}^n Q_{ij} P_j = \min_{w_i \in C_i} \left\{ \frac{\prod_{j=1}^n P_j^{w_{ij}}}{A_i(w_i)} \right\} = K_i(P), \quad (95)$$

yielding that T_i is generated by G_i^* . Finally, Point 2 of Theorem 4 implies that there exists a unique regular and concave production function that generates T_i , so that G_i^* is unique, as desired.

(ii) implies (i). The cost operator generated by G_i is

$$K_i(P) = \min_{w_i \in C_i} \frac{\prod_{j=1}^n P_j^{w_{ij}}}{A_i(w_i)} \quad \forall P \in \mathbb{R}_{++}^n. \quad (96)$$

In turn, this yields that

$$T_i(p) = \min_{w_i \in C_i} \left\{ \langle w_i, p \rangle + \ln \left(\frac{1}{A_i(w_i)} \right) \right\} \quad \forall p \in \mathbb{R}^n, \quad (97)$$

which is a concave function, as desired. $\square$

Proof of Corollary 1. For every sector $i \in [n]$, Proposition 2 and Equation 97 give that

$$T_i(p) = \min_{w_i \in C_i} \left\{ \langle w_i, p \rangle + \ln \left(\frac{1}{A_i(w_i)} \right) \right\} \quad \forall p \in \mathbb{R}^n. \quad (98)$$

By Equation 98, observe that for every $i \in [n]$, T_i can be written as in Equation 71 with $\lambda = 0$ and

$$c_i(\tilde{w}_i) = \begin{cases} \ln \left(\frac{1}{A_i(\tilde{w}_i)} \right) & \text{if } \tilde{w}_i \in C_i, \\ \infty & \text{otherwise.} \end{cases} \quad (99)$$

which is clearly lower semicontinuous. Therefore, the result follows by Lemma 2. $\square$

Proof of Proposition 3. It follows directly from Proposition 12. $\square$

Proof of Corollary 2. The proof follows from an entirely symmetric argument to that of Corollary 1. $\square$

Proof of Proposition 4. First, assume that sector i is resilient. Fix $j \in [n]$ and observe that by Definition 3.1, Proposition 5.7, and Theorem 7.4 in Dal Maso (1993) we have:

$$\lim_{\lambda \rightarrow \infty} \frac{T_i(\lambda e_j)}{\lambda} = \lim_{\lambda \rightarrow \infty} \min_{w_i \in C_i} \left\{ w_{ij} + \frac{1}{\lambda} \ln \left(\frac{1}{A_i(w_i)} \right) \right\} = \min_{w_i \in C_i^R} w_{ij} \quad (100)$$

With this, sector i can abate shocks from sector j if and only if there exists $w_i \in C_i^R$ with $w_{ij} = 0$. Point 2 on the fragile-sector case follows from an entirely symmetric argument. $\square$

Proof of Proposition 5. 1. It follows by Theorem 14 in Marinacci and Montrucchio (2008). $\square$

2. Immediate. $\square$

C Proofs of the Results in Section 6

We say that a function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is *regular at x* if and only if

$$\lim_{h \downarrow 0} \frac{g(x + hd) - g(x)}{h} = \liminf_{x' \rightarrow x, h \downarrow 0} \frac{g(x' + hd) - g(x')}{h} \quad \forall d \in \mathbb{R}^n. \quad (101)$$

We say that g is *regular* if and only if it is regular at each point of its domain. Define $M : \Delta \times \mathbb{R}^n \rightarrow \mathbb{R}$ by $M(\alpha, x) = \ln(\sum_{i=1}^n e^{x_i \alpha_i})$ for all $\alpha \in \Delta$ and for all $x \in \mathbb{R}^n$. We define $\mathcal{M} : \mathcal{W} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $\mathcal{M}_i(W, x) = M(w_i, x)$ for all $W \in \mathcal{W}$, for all $x \in \mathbb{R}^n$, and for all $i \in [n]$.

By Proposition 1, for every $i \in [n]$, if T_i is generated by a production function G_i then $T_i(x) = \min_{\alpha \in \Delta} M(\alpha, x) + c(\alpha)$ where $c : \Delta \rightarrow [0, \infty]$ is grounded (i.e., $\min_{\alpha \in \Delta} c(\alpha) = 0$)

and lower semicontinuous. In that case, define by $\sigma_{T_i}(x)$ the subset of α s in Δ such that $T_i(x) = M(\alpha, x) + c(\alpha)$. It easily follows that $\sigma_{T_i}(x)$ is nonempty and compact for all $x \in \mathbb{R}^n$.

Remark 5. We list here some notation and make a few observations. Given an operator $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$, we denote the Jacobian of S at x by $J_S(x)$. If S is normalized, non-decreasing, and translation invariant and admits a Jacobian at x , then $J_S(x)$ is a stochastic matrix. We endow the space of $n \times n$ matrices with the norm $\| \cdot \|_* : M_n(\mathbb{R}) \rightarrow [0, \infty)$ defined by

$$\|W\|_* = \max_{i \in [n]} \sum_{j=1}^n |w_{ij}| \quad \forall W \in M_n(\mathbb{R}). \quad (102)$$

This is the operator norm when W is viewed as a linear operator from $\mathbb{R}^n$ to $\mathbb{R}^n$ and $\mathbb{R}^n$ is endowed with the supnorm $\| \cdot \|_\infty$. Given $\beta \in (0, 1)$, define $B : \mathcal{W} \rightarrow \mathcal{W}$ by $B(W) = \Omega_W$ for all $W \in \mathcal{W}$. It is standard to show that B is a Lipschitz continuous operator with respect to the norm $\| \cdot \|_*$ and with constant $(1 - \beta) \sum_{t=0}^{\infty} \beta^t$. Finally, B is injective. By [Horn and Johnson \(2012\)](#), Theorem 8.3.4 and Problem 5.6.P26, observe that for each $W \in \mathcal{W}$ the spectral radius of βW is $\beta < 1$ and, in particular, $(I - \beta W)^{-1} = \sum_{t=0}^{\infty} \beta^t W^t$. We can conclude that $B(W) = B(\hat{W})$ implies that $(I - \beta W)^{-1} = (I - \beta \hat{W})^{-1}$, yielding that $I - \beta \hat{W} = I - \beta W$, that is, $W = \hat{W}$, proving injectivity. $\triangle$

Lemma 3. *Let $i \in [n]$. If T_i is generated by a production function G_i then T_i is regular.*

Proof. Fix $x \in \mathbb{R}^n$. Since K_i is concave and locally Lipschitz, we have that $-K_i$ is locally Lipschitz and regular a la Clarke at $\exp(x)$ (see [Clarke, 1990](#), Definition 2.3.4 and Proposition 2.3.6). By [Clarke \(1990\)](#), p. 32 and Theorem 2.3.10, and since $\exp$ is continuously differentiable, it follows that $-K_i \circ \exp$ is regular a la Clarke at x . Define $\gamma : \mathbb{R}_{--} \rightarrow \mathbb{R}$ by $\gamma(s) = -\ln(-s)$ for all $s \in \mathbb{R}_{--}$. By [Clarke \(1990\)](#), p. 32 and Theorem 2.3.10, and since γ is continuously differentiable, we have that γ is regular a la Clarke. By [Clarke \(1990\)](#), point (i) of Theorem 2.3.9, and since γ is non-decreasing and $-K_i \circ \exp$ is regular a la Clarke at x , this implies that $\gamma \circ -K_i \circ \exp$ is regular a la Clarke at x . It is immediate to see that $T_i = -\gamma \circ (-K_i \circ \exp)$, proving that T_i is regular in our sense. $\square$

Lemma 4. *Let $i \in [n]$. If T_i is generated by a production function G_i then for each $x, d \in \mathbb{R}^n$*

$$\partial_C T_i(x) = \text{co} \{ \nabla M(w_i, x) : w_i \in \sigma_{T_i}(x) \} \quad \text{and} \quad \lim_{h \downarrow 0} \frac{T_i(x + hd) - T_i(x)}{h} = \min_{w_i \in \sigma_{T_i}(x)} \langle \nabla M(w_i, x), d \rangle. \quad (103)$$

Proof. The result follows from [Clarke \(2013\)](#), Theorem 10.22, whose assumptions are satisfied by Lemma 3. $\square$

Corollary 3. *Fix $i \in [n]$. If T_i is generated by a production function G_i , then for each $y, z \in \mathbb{R}^n$ there exist $\lambda_{y,z} \in [0, 1]$ and $w_i \in \sigma_{T_i}(\lambda_{y,z}z + (1 - \lambda_{y,z})y)$ such that*

$$T_i(z) - T_i(y) \geq \langle \nabla M(w_i, \lambda_{y,z}z + (1 - \lambda_{y,z})y), z - y \rangle. \quad (104)$$

Proof. Fix $y, z \in \mathbb{R}^n$. By Lebourg's Mean Value Theorem (see, e.g., [Clarke, 1990](#), Theorem 2.3.7) and since T_i is Lipschitz continuous, there exists $\lambda_{y,z} \in [0, 1]$ such that

$$T_i(z) - T_i(y) \geq \min_{w_i \in \partial_C T_i(\lambda_{y,z}z + (1-\lambda_{y,z})y)} \langle w_i, z - y \rangle. \quad (105)$$

By Lemma 4, we have that

$$\begin{aligned} T_i(z) - T_i(y) &\geq \min_{w_i \in \partial_C T_i(\lambda_{y,z}z + (1-\lambda_{y,z})y)} \langle w_i, z - y \rangle \\ &= \min_{w_i \in \sigma_{T_i}(\lambda_{y,z}z + (1-\lambda_{y,z})y)} \langle \nabla M(w_i, \lambda_{y,z}z + (1-\lambda_{y,z})y), z - y \rangle. \end{aligned} \quad (106)$$

By choosing $w_i \in \sigma_{T_i}(\lambda_{y,z}z + (1-\lambda_{y,z})y)$ which achieves the min above, the statement follows. $\square$

Lemma 5. Let $W \in \mathcal{W}$, $\beta \in (0, 1)$, and $d \in \mathbb{R}^n$. If $z \in \mathbb{R}^n$ is such that $z \leq (1-\beta)d + \beta Wz$ (resp. $\geq$), then $z \leq \Omega_W d$ (resp. $\geq$).

Proof. We prove by induction that

$$z \leq (1-\beta) \sum_{t=0}^{\tau} \beta^t W^t d + \beta^{\tau+1} W^{\tau+1} z \quad (\text{resp. } \geq) \quad \forall \tau \in \mathbb{N}_0. \quad (107)$$

Initial Step. $\tau = 0$. By assumption, we have that

$$z \leq (1-\beta)d + \beta Wz = (1-\beta) \sum_{t=0}^{\tau} \beta^t W^t d + \beta^{\tau+1} W^{\tau+1} z \quad (\text{resp. } \geq), \quad (108)$$

proving the initial step.

Inductive Step. We assume the inequality is true for τ and prove it holds for $\tau + 1$. By assumption, we have that

$$z \leq (1-\beta) \sum_{t=0}^{\tau} \beta^t W^t d + \beta^{\tau+1} W^{\tau+1} z \leq (1-\beta) \sum_{t=0}^{\tau} \beta^t W^t d + \beta^{\tau+1} W^{\tau+1} ((1-\beta)d + \beta Wz) \quad (109)$$

$$= (1-\beta) \sum_{t=0}^{\tau+1} \beta^t W^t d + \beta^{\tau+2} W^{\tau+2} z, \quad (110)$$

proving the inductive step.

The inequality (107) follows by induction. By passing to the limit in τ and since W is a stochastic matrix, we have that $z \leq \Omega_W d$ (resp. $\geq$). $\square$

Given $x, d \in \mathbb{R}^n$ and $h > 0$, define $q_{h,x,d} = \frac{\Pi(x+hd) - \Pi(x)}{h}$. Given $x, d \in \mathbb{R}^n$, since Π is Lipschitz continuous of order 1, we have that $\{q_{h,x,d}\}_{h>0} \subseteq [-\|d\|_\infty, \|d\|_\infty]^n$.

Lemma 6. Let T be generated by a profile of production functions G and $\beta \in (0, 1)$. If $x, d \in \mathbb{R}^n$ and $\tilde{W} \in \Sigma(\Pi(x))$ and $\{h_k\}_{k \in \mathbb{N}} \subseteq (0, \infty)$ is such that $\lim_k h_k = 0$ and $\lim_k q_{h_k, x, d} = q$, then

$$q \leq \Omega_{J_{\mathcal{M}(\tilde{W}, \cdot)}(\Pi(x))} d. \quad (111)$$

Proof. Fix $x, d \in \mathbb{R}^n$. Since we fixed x and d , we denote $q_{h, x, d}$ simply by q_h for all $h > 0$. Consider $\tilde{W} \in \Sigma(\Pi(x))$. Note that, by Proposition 1, for each $x \in \mathbb{R}^n$

$$T(y) - T(x) \leq \mathcal{M}(\tilde{W}, y) - \mathcal{M}(\tilde{W}, x) \quad \forall \tilde{W} \in \Sigma(x), \forall y \in \mathbb{R}^n. \quad (112)$$

Therefore, we have that

$$\Pi(x + hd) - \Pi(x) = (1 - \beta)(x + hd) + \beta T(\Pi(x + hd)) - (1 - \beta)x - \beta T(\Pi(x)) \quad (113)$$

$$= (1 - \beta)hd + \beta [T(\Pi(x + hd)) - T(\Pi(x))] \quad (114)$$

$$\leq (1 - \beta)hd + \beta \left(\mathcal{M}(\tilde{W}, \Pi(x + hd)) - \mathcal{M}(\tilde{W}, \Pi(x)) \right) \quad \forall h > 0. \quad (115)$$

By the Mean Value Theorem, we have that for each $h > 0$ and for each $i \in [n]$ there exists $\tau_{i, h} \in [0, 1]$ such that

$$\mathcal{M}_i(\tilde{W}, \Pi(x + hd)) - \mathcal{M}_i(\tilde{W}, \Pi(x)) \quad (116)$$

$$= M(\tilde{w}_i, \Pi(x + hd)) - M(\tilde{w}_i, \Pi(x)) \quad (117)$$

$$= \langle \nabla M(\tilde{w}_i, \tau_{i, h} \Pi(x + hd) + (1 - \tau_{i, h}) \Pi(x)), \Pi(x + hd) - \Pi(x) \rangle. \quad (118)$$

For each $h > 0$ let W_h be the matrix whose i -th row is $\nabla M(\tilde{w}_i, \tau_{i, h} \Pi(x + hd) + (1 - \tau_{i, h}) \Pi(x))$. Note that

$$\mathcal{M}(\tilde{W}, \Pi(x + hd)) - \mathcal{M}(\tilde{W}, \Pi(x)) = W_h (\Pi(x + hd) - \Pi(x)) \quad \forall h > 0. \quad (119)$$

Since Π is Lipschitz continuous of order 1 and $(\omega, z) \mapsto \nabla M(\omega, z)$ is continuous on $\Delta \times \mathbb{R}^n$, we have that $\lim_{h \downarrow 0} W_h = J_{\mathcal{M}(\tilde{W}, \cdot)}(\Pi(x))$. Since $h > 0$, it follows that

$$\frac{\Pi(x + hd) - \Pi(x)}{h} \leq (1 - \beta)d + \beta \left(\frac{\mathcal{M}(\tilde{W}, \Pi(x + hd)) - \mathcal{M}(\tilde{W}, \Pi(x))}{h} \right) \quad (120)$$

$$= (1 - \beta)d + \beta W_h \frac{\Pi(x + hd) - \Pi(x)}{h} \quad \forall h > 0, \quad (121)$$

that is,

$$q_h \leq (1 - \beta)d + \beta W_h q_h \quad \forall h > 0. \quad (122)$$

By Lemma 5 and since h was arbitrarily chosen and W_h is a stochastic matrix, we have that $q_h \leq \Omega_{W_h} d$ for all $h > 0$. By Remark 5 and since $\lim_k W_{h_k} = J_{\mathcal{M}(\tilde{W}, \cdot)}(\Pi(x))$, $\lim_k q_{h_k} = q$, and $q_{h_k} \leq \Omega_{W_{h_k}} d$ for all $k \in \mathbb{N}$, it follows that $q \leq \Omega_{J_{\mathcal{M}(\tilde{W}, \cdot)}(\Pi(x))} d$, proving (111). $\square$

Lemma 7. Let T be generated by a profile of production functions G and $\beta \in (0, 1)$. If $x, d \in \mathbb{R}^n$ and $\{h_k\}_{k \in \mathbb{N}} \subseteq (0, \infty)$ is such that $\lim_k h_k = 0$ and $\lim_k q_{h_k, x, d} = q$, then there exists $\hat{W} \in \Sigma(\Pi(x))$ such that

$$q \geq \Omega_{J_{\mathcal{M}(\hat{W}, \cdot)}(\Pi(x))} d. \quad (123)$$

In particular, $q = \min_{\tilde{W} \in \Sigma(\Pi(x))} \Omega_{J_{\mathcal{M}(\tilde{W}, \cdot)}(\Pi(x))} d$.

Proof. Fix $x, d \in \mathbb{R}^n$. Since we fixed x and d , we denote $q_{h, x, d}$ by simply q_h for all $h > 0$. By Corollary 3, for each $h > 0$ and for each $i \in [n]$ there exist $\lambda_{i, h} \stackrel{\text{def}}{=} \lambda_{i, \Pi(x), \Pi(x+hd)} \in [0, 1]$ and $\hat{w}_{i, h} \in \sigma_{T_i}(\lambda_{i, h} \Pi(x+hd) + (1 - \lambda_{i, h}) \Pi(x))$ such that

$$T_i(\Pi(x+hd)) - T_i(\Pi(x)) \geq \langle \nabla M(\hat{w}_{i, h}, \lambda_{i, h} \Pi(x+hd) + (1 - \lambda_{i, h}) \Pi(x)), \Pi(x+hd) - \Pi(x) \rangle. \quad (124)$$

Let $\hat{W}_h$ be the matrix whose i -th row is $\hat{w}_{i, h}$ and define W_h as the matrix whose i -th row is $\nabla M(\hat{w}_{i, h}, \lambda_{i, h} \Pi(x+hd) + (1 - \lambda_{i, h}) \Pi(x))$. This implies that

$$\Pi(x+hd) - \Pi(x) = (1 - \beta)(x+hd) + \beta T(\Pi(x+hd)) - (1 - \beta)x - \beta T(\Pi(x)) \quad (125)$$

$$= (1 - \beta)hd + \beta [T(\Pi(x+hd)) - T(\Pi(x))] \quad (126)$$

$$\geq (1 - \beta)hd + \beta W_h(\Pi(x+hd) - \Pi(x)). \quad (127)$$

By dividing by h , we obtain that

$$q_h \geq (1 - \beta)d + \beta W_h q_h \quad \forall h > 0. \quad (128)$$

By Lemma 5 and since h was arbitrarily chosen and W_h is a stochastic matrix, we have that $q_h \geq \Omega_{W_h} d$ for all $h > 0$. Consider now $\{h_k\}_{k \in \mathbb{N}} \subseteq (0, \infty)$ such that $\lim_k h_k = 0$ and $\lim_k q_{h_k} = q$. Since $\{\hat{W}_{h_k}\}_{k \in \mathbb{N}} \subseteq \mathcal{W}$, we have that there exists a subsequence $\{h_{k_l}\}_{l \in \mathbb{N}} \subseteq \{h_k\}_{k \in \mathbb{N}}$ such that $\lim_l \hat{W}_{h_{k_l}} = \hat{W}$ and, in particular, $\lim_l \hat{w}_{i, h_{k_l}} = \hat{w}_i$ for all $i \in [n]$. Since Π is Lipschitz continuous, we have that $\hat{w}_i \in \sigma_{T_i}(\Pi(x))$ for all $i \in [n]$ and, in particular, $\hat{W} \in \Sigma(\Pi(x))$. Since Π is Lipschitz continuous and $(\omega, z) \mapsto \nabla M(\omega, z)$ is continuous on $\Delta \times \mathbb{R}^n$, we have that

$$\lim_l \nabla M(\hat{w}_{i, h_{k_l}}, \lambda_{i, h_{k_l}} \Pi(x + h_{k_l} d) + (1 - \lambda_{i, h_{k_l}}) \Pi(x)) = \nabla M(\hat{w}_i, \Pi(x)) \quad \forall i \in [n].$$

This implies that $\lim_l W_{h_{k_l}} = J_{\mathcal{M}(\hat{W}, \cdot)}(\Pi(x))$ with $\hat{W} \in \Sigma(\Pi(x))$. By Remark 5 and since $\lim_l W_{h_{k_l}} = J_{\mathcal{M}(\hat{W}, \cdot)}(\Pi(x))$, $\lim_l q_{h_{k_l}} = q$, and $q_{h_{k_l}} \geq \Omega_{W_{h_{k_l}}} d$ for all $l \in \mathbb{N}$, it follows that $q \geq \Omega_{J_{\mathcal{M}(\hat{W}, \cdot)}(\Pi(x))} d$ with $\hat{W} \in \Sigma(\Pi(x))$, proving the first part of the statement. By Lemma 6, we have that $\Omega_{J_{\mathcal{M}(\hat{W}, \cdot)}(\Pi(x))} d \geq q$ and, in particular,

$$q = \Omega_{J_{\mathcal{M}(\hat{W}, \cdot)}(\Pi(x))} d \leq \Omega_{J_{\mathcal{M}(\tilde{W}, \cdot)}(\Pi(x))} d \quad \forall \tilde{W} \in \Sigma(\Pi(x)), \quad (129)$$

proving that $q = \min_{\tilde{W} \in \Sigma(\Pi(x))} \Omega_{J_{\mathcal{M}(\tilde{W}, \cdot)}(\Pi(x))} d$. $\square$

Proof of Theorem 2. Fix $x, d \in \mathbb{R}^n$. Consider a sequence $\{h_k\}_{k \in \mathbb{N}} \subseteq (0, \infty)$ such that $\lim_k h_k = 0$. Since we fixed x and d , we denote $q_{h, x, d}$ simply by q_h for all $h > 0$. As noted, $\{q_{h_k}\}_{k \in \mathbb{N}}$ is a

bounded sequence. Consider a subsequence $\{q_{h_{k_l}}\}_{l \in \mathbb{N}} \subseteq \{q_{h_k}\}_{k \in \mathbb{N}}$. Since $\{q_{h_k}\}_{k \in \mathbb{N}}$ is a bounded sequence, $\{q_{h_{k_l}}\}_{l \in \mathbb{N}}$ has a further subsequence $\{q_{h_{k_{l(m)}}}\}_{m \in \mathbb{N}}$ that converges.

By Lemma 7, it converges to $\min_{\tilde{W} \in \Sigma(\Pi(x))} \Omega_{J_{\mathcal{M}(\tilde{W}, \cdot)}(\Pi(x))} d$. By (Carothers, 2000, p. 47, Exercise 39) we can conclude that

$$\lim_k \frac{\Pi(x + h_k d) - \Pi(x)}{h_k} = \min_{\tilde{W} \in \Sigma(\Pi(x))} \Omega_{L(\tilde{W}, \Pi(x))} d.$$

Since $\{h_k\}_{k \in \mathbb{N}}$ and d were arbitrarily chosen, the statement follows. $\square$

Corollary 4. Fix $i \in [n]$ and let T_i be generated by a production function G_i . The following statements are equivalent:

- (i) T_i is Gateaux differentiable at $x \in \mathbb{R}^n$;
- (ii) $\sigma_{T_i}(x)$ is a singleton.

Proof. Fix $x \in \mathbb{R}^n$. Define $T_{i,\circ}(x; \cdot) : \mathbb{R}^n \rightarrow \mathbb{R}$ by $T_{i,\circ}(x; d) = \lim_{h \downarrow 0} \frac{T_i(x+hd) - T_i(x)}{h}$ for all $d \in \mathbb{R}^n$. By Lemma 4, it follows that $T_{i,\circ}(x; \cdot) = \min_{w_i \in \sigma_{T_i}(x)} \langle \nabla M(w_i, x), d \rangle$.

(i) implies (ii). Since T_i is Gateaux differentiable at $x \in \mathbb{R}^n$, $T_{i,\circ}(x; \cdot)$ is linear and, in particular, $\{\nabla M(w_i, x) : w_i \in \sigma_{T_i}(x)\}$ is a singleton. Since if $w_i \neq w'_i$ then $\nabla M(w_i, x) \neq \nabla M(w'_i, x)$, we can conclude that $\sigma_{T_i}(x)$ is a singleton.

(ii) implies (i). If $\sigma_{T_i}(x)$ is a singleton, that is, $\sigma_{T_i}(x) = \{w_i\}$, then $T_{i,\circ}(x; \cdot) = \langle \nabla M(w_i, x), \cdot \rangle$, proving the statement. $\square$

Lemma 8. If $i \in [n]$ and $P \in \mathbb{R}_{++}^n$, then the following equation holds:

$$\min_{\substack{(Q_{i0}, Q_i) \in \mathbb{R}_+^{n+1} \\ F_i(Q_{i0}, Q_i) \geq 1}} \sum_{j=1}^n Q_{ij} P_j + Q_{i0} = \frac{1}{Z_i} K_i(P)^\beta \quad (130)$$

Moreover, we have that:

- (i) If $(\hat{Q}_{i0}, \hat{Q}_i) \in \mathbb{R}_+^{n+1}$ solves (130), then $\hat{Q}_{i0} = \frac{1-\beta}{Z_i} K_i(P)^\beta$ and there exists $\gamma > 0$ such that $(\gamma \hat{Q}_{i1}, \dots, \gamma \hat{Q}_{in})$ solves (12).
- (ii) If $Q_i \in \mathbb{R}_+^n$ solves (12), then there exist $\gamma > 0$ and $\hat{Q}_{i0} > 0$ such that $(\hat{Q}_{i0}, \gamma Q_{i1}, \dots, \gamma Q_{in}) \in \mathbb{R}_+^{n+1}$ solves (130).
- (iii) If $(\hat{Q}_{i0}, \hat{Q}_i) \in \mathbb{R}_+^{n+1}$ solves (130) and $P_i = \min_{(Q_{i0}, Q_i) \in \mathbb{R}_+^{n+1} : F_i(Q_{i0}, Q_i) \geq 1} \sum_{j=1}^n Q_{ij} P_j + Q_{i0}$, then

$$\beta = \frac{\sum_{j=1}^n \hat{Q}_{ij} P_j}{P_i}. \quad (131)$$

Proof. We only prove that the minima in Equations 12 and 130 are attained. Given this, Equation 130 and points (i)-(iii) are routine and follow from some tedious algebra. Fix $P \in \mathbb{R}_{++}^n$ and $i \in [n]$. Observe first that the minimized function in Equation 12 is continuous, and the set over which the minimization is performed is nonempty and includes some $\hat{Q}_i \in \Delta$. The

set over which the minimization is performed is closed by the continuity of G_i . Moreover, for any $P \in \mathbb{R}_{++}^n$ the same value is obtained if we restrict the set over which the minimization is performed to $\{Q_i : G_i(Q_i) \geq 1 \text{ and } P_j Q_{ij} \leq \sum_{k=1}^n P_k \hat{Q}_{ik}, \forall j \in [n]\}$ which is also compact, and the minimum is thus attained there. A similar argument shows that the minimum in Equation 130 is attained. $\square$

Lemma 9. *Let $i \in [n]$ and $P \in \mathbb{R}_{++}^n$. If K_i is generated by a production function G_i , we have that:*

- (i) *If $Q_i \in \mathbb{R}_+^n$ solves (12), then $\frac{1}{\sum_{j=1}^n Q_{ij}} Q_i \in \Delta$ solves the right-hand side problem of (26).*
- (ii) *If $\tilde{w}_i \in \Delta$ solves the right-hand side problem of (26), then there exists $\gamma > 0$ such that $\gamma \tilde{w}_i \in \mathbb{R}_+^n$ solves (12).*

Proof. (i). Assume $Q_i \in \mathbb{R}_+^n$ solves (12). Define $w_i = \frac{1}{\sum_{j=1}^n Q_{ij}} Q_i \in \Delta$. Since G_i is positive homogeneous and $G_i(Q_i) \geq 1$, observe that

$$K_i(P) = \sum_{j=1}^n Q_{ij} P_j \geq \frac{\sum_{j=1}^n Q_{ij} P_j}{G_i(Q_i)} = \frac{\sum_{j=1}^n w_{ij} P_j}{G_i(w_i)} \geq K_i(P), \quad (132)$$

proving that w_i solves the right-hand side problem of (26).

(ii). Assume that w_i solves the right-hand side problem of (26). It follows that $G_i(w_i) > 0$. Consider $\gamma = G_i(w_i)^{-1}$ and define $Q_i = \gamma w_i$. Since G_i is positive homogeneous, we have that $G_i(Q_i) = 1$ and

$$K_i(P) = \min_{\tilde{w}_i \in \Delta} \frac{\sum_{j=1}^n \tilde{w}_{ij} P_j}{G_i(\tilde{w}_i)} = \frac{\sum_{j=1}^n w_{ij} P_j}{G_i(w_i)} = \frac{\sum_{j=1}^n \gamma w_{ij} P_j}{G_i(\gamma w_i)} = \frac{\sum_{j=1}^n Q_{ij} P_j}{G_i(Q_i)} = \sum_{j=1}^n Q_{ij} P_j \geq K_i(P), \quad (133)$$

proving that $Q_i = \gamma w_i$ solves (12). $\square$

Corollary 5. *Let $i \in [n]$ and $P \in \mathbb{R}_{++}^n$. The following statements are true:*

- (i) *If $\hat{Q}_i \in \mathbb{R}_+^{n+1}$ solves (130), then $\frac{1}{\sum_{j=1}^n \hat{Q}_{ij}} (\hat{Q}_{i1}, \dots, \hat{Q}_{in})$ solves the right-hand side problem of (26).*
- (ii) *If $w_i \in \Delta$ solves the right-hand side problem of (26), then there exist $\gamma > 0$ and $\hat{Q}_{i0} > 0$ such that $\hat{Q}_i = (\hat{Q}_{i0}, \gamma w_{i1}, \dots, \gamma w_{in}) \in \mathbb{R}_+^{n+1}$ solves (130).*

Proof. (i). Fix $i \in [n]$ and $P \in \mathbb{R}_{++}^n$. By Lemma 8, we have that there exists $\gamma > 0$ such that $(\gamma \hat{Q}_{i1}, \dots, \gamma \hat{Q}_{in})$ solves (12). By Lemma 9, it follows that $\frac{1}{\sum_{j=1}^n \gamma \hat{Q}_{ij}} (\gamma \hat{Q}_{i1}, \dots, \gamma \hat{Q}_{in}) = \frac{1}{\sum_{j=1}^n \hat{Q}_{ij}} (\hat{Q}_{i1}, \dots, \hat{Q}_{in})$ solves the right-hand side problem of (26).

(ii). By Lemma 9, if $w_i \in \Delta$ solves the right-hand side problem of (26), then there exists $\gamma_1 > 0$ such that $\gamma_1 w_i$ solves (12). By Lemma 8, there exist $\gamma_2 > 0$ and $\hat{Q}_{i0} > 0$ such that $\hat{Q}_i = (\hat{Q}_{i0}, \gamma_2 \gamma_1 w_{i1}, \dots, \gamma_2 \gamma_1 w_{in}) \in \mathbb{R}_+^{n+1}$ solves (130). $\square$

We denote the set of all competitive equilibria by $\mathfrak{E}$. Given $\mathcal{E} \in \mathfrak{E}$, that is a tuple $(\bar{P}, \hat{Q}, \bar{Y}, \bar{C})$, we define $E_{\mathcal{E}}$ as the $n \times n$ matrix whose ij -th entry is $\frac{\hat{Q}_{ij}\bar{P}_j}{F_i(\hat{Q}_i)\bar{P}_i}$. We also define $\tilde{W}_{\mathcal{E}}$ as the matrix whose ij -th entry is $\frac{\hat{Q}_{ij}}{\sum_{k=1}^n \hat{Q}_{ik}}$.

Proposition 8. *We have that:*

- (i) $\mathfrak{E}$ is nonempty.
- (ii) $\left\{ \tilde{W}_{\mathcal{E}} \right\}_{\mathcal{E} \in \mathfrak{E}} = \Sigma(\Pi(\bar{x}))$ and $\left\{ L\left(\tilde{W}_{(\bar{P}, \hat{Q}, \bar{Y}, \bar{C})}, \ln(\bar{P})\right) \right\}_{(\bar{P}, \hat{Q}, \bar{Y}, \bar{C}) \in \mathfrak{E}} = \left\{ J_{\mathcal{M}(\tilde{W}, \cdot)}(\Pi(\bar{x})) : \tilde{W} \in \Sigma(\Pi(\bar{x})) \right\}$.
- (iii) If $\mathcal{E} \in \mathfrak{E}$, then $\lambda_{(\bar{P}, \hat{Q}, \bar{Y}, \bar{C})}^T = \xi^T \left(I - \beta L\left(\tilde{W}_{(\bar{P}, \hat{Q}, \bar{Y}, \bar{C})}, \ln(\bar{P})\right) \right)^{-1}$.
- (iv) If $(\tilde{P}, \tilde{Q}, \tilde{Y}, \tilde{C}), (\bar{P}, \hat{Q}, \bar{Y}, \bar{C}) \in \mathfrak{E}$, then $\tilde{P} = \bar{P}$.

Proof. (i). Consider the (unique) equilibrium price $\bar{P}$ and a matrix $\hat{Q}$ such that $(\hat{Q}_{i0}, \hat{Q}_i)$ solves Equation 130 for all $i \in [n]$ with $P = \bar{P}$. Compute $\bar{C} \in \operatorname{argmax}_{C \in \mathbb{R}_+^n: \sum_{i=1}^n \bar{P}_i C_i \leq 1} U(C)$. Next, consider the equation $\lambda^T = \bar{D}^T + \lambda^T E_{\bar{P}, \hat{Q}}$ with unknown $\lambda \in \mathbb{R}^n$ where $\bar{D}$ is the vector whose j -th entry is $\bar{C}_j \bar{P}_j$. By point (iii) of Lemma 8 and since $\bar{P}$ is an equilibrium price, we have that $\beta = \sum_{j=1}^n \hat{Q}_{ij} \bar{P}_j / \bar{P}_i$ for all $i \in [n]$. Since $F_i(\hat{Q}_i) = 1$ for all $i \in [n]$, this implies that

$$\frac{\hat{Q}_{ij} \bar{P}_j}{\bar{P}_i} = \frac{\sum_{k=1}^n \hat{Q}_{ik} \bar{P}_k}{\bar{P}_i} \frac{\hat{Q}_{ij} \bar{P}_j}{\sum_{k=1}^n \hat{Q}_{ik} \bar{P}_k} = \beta \frac{\hat{Q}_{ij} \bar{P}_j}{\sum_{k=1}^n \hat{Q}_{ik} \bar{P}_k} \quad \forall i, j \in [n], \quad (134)$$

that is $E_{\bar{P}, \hat{Q}} = \beta L\left(\tilde{W}_{(\bar{P}, \hat{Q}, \bar{Y}, \bar{C})}, \ln(\bar{P})\right)$, yielding that $\lambda^T = \bar{D}^T + \lambda^T \beta L\left(\tilde{W}_{(\bar{P}, \hat{Q}, \bar{Y}, \bar{C})}, \ln(\bar{P})\right)$. Since $L\left(\tilde{W}_{(\bar{P}, \hat{Q}, \bar{Y}, \bar{C})}, \ln(\bar{P})\right)$ is a stochastic matrix, $\bar{D} \geq 0$, and $\beta \in (0, 1)$, it follows that this equation has a (unique) solution $\lambda^T = \bar{D}^T \left(I - \beta L\left(\tilde{W}_{(\bar{P}, \hat{Q}, \bar{Y}, \bar{C})}, \ln(\bar{P})\right) \right)^{-1} \geq 0$. Since $\bar{P} \in \mathbb{R}_{++}^n$ and $\lambda \geq 0$, define $\bar{Y} \in \mathbb{R}_+^n$ by $\bar{Y}_i = \lambda_i / \bar{P}_i$ for all $i \in [n]$. Since F_i is positive homogeneous and $F_i(\hat{Q}_i) = 1$ for all $i \in [n]$ and $\lambda^T = \bar{D}^T + \lambda^T E_{\bar{P}, \hat{Q}}$, it follows that for each $j \in [n]$

$$F_j\left(\bar{Y}_j \hat{Q}_{j0}, \bar{Y}_j \hat{Q}_j\right) \bar{P}_j = \bar{Y}_j F_j\left(\hat{Q}_j\right) \bar{P}_j = \lambda_j = \bar{D}_j + \sum_{i=1}^n \lambda_i \frac{\hat{Q}_{ij} \bar{P}_j}{\bar{P}_i} = \bar{C}_j \bar{P}_j + \sum_{i=1}^n \bar{Y}_i \hat{Q}_{ij} \bar{P}_j \quad \forall j \in [n]. \quad (135)$$

By dividing by $\bar{P}_j$ and since $\bar{P}_j > 0$ for all $j \in [n]$, it follows that the market clearing conditions of the n sectors are satisfied. Since $\bar{C}_j \bar{P}_j + \sum_{i=1}^n \bar{Y}_i \hat{Q}_{ij} \bar{P}_j = F_j\left(\bar{Y}_j \hat{Q}_{j0}, \bar{Y}_j \hat{Q}_j\right) \bar{P}_j$ for all $j \in [n]$, we obtain that

$$\sum_{i=1}^n \sum_{j=1}^n \bar{Y}_i \hat{Q}_{ij} \bar{P}_j + \sum_{j=1}^n \bar{C}_j \bar{P}_j = \sum_{j=1}^n \sum_{i=1}^n \bar{Y}_i \hat{Q}_{ij} \bar{P}_j + \sum_{j=1}^n \bar{C}_j \bar{P}_j = \sum_{j=1}^n F_j\left(\bar{Y}_j \hat{Q}_{j0}, \bar{Y}_j \hat{Q}_j\right) \bar{P}_j. \quad (136)$$

Since the preferences of the representative agent are monotone and $\bar{P}_i = \sum_{j=1}^n \hat{Q}_{ij} \bar{P}_j + \hat{Q}_{i0}$ for

all $i \in [n]$, we have that

$$1 = \sum_{j=1}^n \bar{C}_j \bar{P}_j = \sum_{i=1}^n \left(F_i \left(\bar{Y}_i \hat{Q}_{i0}, \bar{Y}_i \hat{Q}_i \right) \bar{P}_i - \sum_{j=1}^n \bar{Y}_i \hat{Q}_{ij} \bar{P}_j \right) = \sum_{i=1}^n \left(\bar{Y}_i \left[F_i \left(\hat{Q}_{i0}, \hat{Q}_i \right) \bar{P}_i - \sum_{j=1}^n \hat{Q}_{ij} \bar{P}_j \right] \right) \quad (137)$$

$$= \sum_{i=1}^n \left(\bar{Y}_i \left[\bar{P}_i - \sum_{j=1}^n \hat{Q}_{ij} \bar{P}_j \right] \right) = \sum_{i=1}^n \bar{Y}_i \hat{Q}_{i0}. \quad (138)$$

This proves that $(\bar{P}, \hat{Q}, \bar{Y}, \bar{C})$ satisfies condition 3 of a competitive equilibrium (cf. Definition 1). By construction of $\bar{P}$ and $\bar{C}$, conditions 2 and 4 of a competitive equilibrium are also satisfied. Finally, since $(\hat{Q}_{i0}, \hat{Q}_i)$ solves (130) for all $i \in [n]$ with $P = \bar{P}$, we have that condition 1 of a competitive equilibrium is satisfied, proving that $(\bar{P}, \hat{Q}, \bar{Y}, \bar{C}) \in \mathfrak{E}$ and the latter set is nonempty.

(ii). Fix an equilibrium $\mathcal{E} = (\bar{P}, \hat{Q}, \bar{Y}, \bar{C})$. It follows that $\bar{P} = \exp \Pi(\bar{x})$. By point (i) of Corollary 5 and since $(\hat{Q}_{i0}, \hat{Q}_i)$ solves (130) for all $i \in [n]$, then $\frac{1}{\sum_{k=1}^n \hat{Q}_{ik}} (\hat{Q}_{i1}, \dots, \hat{Q}_{in})$ solves the right-hand side problem of (26) with $\bar{p} = \Pi(\bar{x})$ for all $i \in [n]$, proving that $\tilde{W}_{\mathcal{E}} \in \Sigma(\Pi(\bar{x}))$. Define $W = J_{\mathcal{M}(\tilde{W}_{\mathcal{E}}, \cdot)}(\Pi(\bar{x}))$. It follows that

$$w_{ij} = \frac{\tilde{w}_{ij} e^{\Pi_j(\bar{x})}}{\sum_{k=1}^n \tilde{w}_{ik} e^{\Pi_k(\bar{x})}} = \frac{\tilde{w}_{ij} \bar{P}_j}{\sum_{k=1}^n \tilde{w}_{ik} \bar{P}_k} = \frac{\frac{\hat{Q}_{ij}}{\sum_{k=1}^n \hat{Q}_{ik}} \bar{P}_j}{\sum_{k=1}^n \frac{\hat{Q}_{ik}}{\sum_{k=1}^n \hat{Q}_{ik}} \bar{P}_k} = \frac{\hat{Q}_{ij} \bar{P}_j}{\sum_{k=1}^n \hat{Q}_{ik} \bar{P}_k}, \quad (139)$$

proving that $W = L(\tilde{W}_{(\bar{P}, \hat{Q}, \bar{Y}, \bar{C})}, \ln(\bar{P}))$. We can conclude that:

$$\left\{ L(\tilde{W}_{(\bar{P}, \hat{Q}, \bar{Y}, \bar{C})}, \ln(\bar{P})) \right\}_{(\bar{P}, \hat{Q}, \bar{Y}, \bar{C}) \in \mathfrak{E}} \subseteq \left\{ J_{\mathcal{M}(\tilde{W}, \cdot)}(\Pi(\bar{x})) : \tilde{W} \in \Sigma(\Pi(\bar{x})) \right\} \quad (140)$$

Consider $\tilde{W} \in \Sigma(\Pi(\bar{x}))$. By point (ii) of Corollary 5, it follows that there exist (unique) $\gamma_i > 0$ and (unique) $\hat{Q}_{i0} > 0$ such that $(\hat{Q}_{i0}, \gamma_i \tilde{w}_{i1}, \dots, \gamma_i \tilde{w}_{in}) \in \mathbb{R}_+^{n+1}$ solves (130) for all $i \in [n]$ with $P = \bar{P} = \exp \Pi(\bar{x})$. By part 1 of this proof, there exist (unique) $\bar{Y} \in \mathbb{R}_+^n$ and $\bar{C} \in \mathbb{R}_+^n$ such that $(\bar{P}, \hat{Q}, \bar{Y}, \bar{C}) \in \mathfrak{E}$. Finally, since $\tilde{W}$ is a stochastic matrix, observe that

$$\frac{\hat{Q}_{ij}}{\sum_{k=1}^n \hat{Q}_{ik}} = \frac{\gamma_i \tilde{w}_{ij}}{\sum_{k=1}^n \gamma_i \tilde{w}_{ik}} = \tilde{w}_{ij} \quad \forall i, j \in [n], \quad (141)$$

proving that $\tilde{W} \in \left\{ \tilde{W}_{\mathcal{E}} \right\}_{\mathcal{E} \in \mathfrak{E}}$ and, in particular, $\left\{ \tilde{W}_{\mathcal{E}} \right\}_{\mathcal{E} \in \mathfrak{E}} \supseteq \Sigma(\Pi(\bar{x}))$. Finally, consider $W \in \left\{ J_{\mathcal{M}(\tilde{W}, \cdot)}(\Pi(\bar{x})) : \tilde{W} \in \Sigma(\Pi(\bar{x})) \right\}$, that is, $W = J_{\mathcal{M}(\tilde{W}, \cdot)}(\Pi(\bar{x}))$ for some $\tilde{W} \in \Sigma(\Pi(\bar{x}))$. By (139) and since $\left\{ \tilde{W}_{\mathcal{E}} \right\}_{\mathcal{E} \in \mathfrak{E}} = \Sigma(\Pi(\bar{x}))$, we have that $\tilde{W} = \tilde{W}_{\mathcal{E}}$ for some $\mathcal{E} = (\bar{P}, \hat{Q}, \bar{Y}, \bar{C}) \in \mathfrak{E}$

and $W = L\left(\tilde{W}_{(\bar{P}, \hat{Q}, \bar{Y}, \bar{C})}, \ln(\bar{P})\right)$, proving that

$$\left\{L\left(\tilde{W}_{(\bar{P}, \hat{Q}, \bar{Y}, \bar{C})}, \ln(\bar{P})\right)\right\}_{(\bar{P}, \hat{Q}, \bar{Y}, \bar{C}) \in \mathfrak{E}} \supseteq \left\{J_{\mathcal{M}(\tilde{W}, \cdot)}(\Pi(\bar{x})) : \tilde{W} \in \Sigma(\Pi(\bar{x}))\right\}. \quad (142)$$

(iii). Fix an equilibrium $\mathcal{E} = (\bar{P}, \hat{Q}, \bar{Y}, \bar{C})$. It follows that

$$\bar{Y}_j \bar{P}_j = \bar{C}_j \bar{P}_j + \sum_{i=1}^n \bar{Y}_i \hat{Q}_{ij} \bar{P}_j = \bar{D}_j + \sum_{i=1}^n (\bar{Y}_i \bar{P}_i) \frac{\hat{Q}_{ij} \bar{P}_j}{\bar{P}_i} \quad \forall j \in [n], \quad (143)$$

that is, $\lambda_{\mathcal{E}}^T = \bar{D}^T + \lambda_{\mathcal{E}}^T E_{\mathcal{E}}$. By the same arguments leading to (134), we can conclude that $\lambda_{\mathcal{E}}^T = \bar{D}^T + \lambda_{\mathcal{E}}^T \beta L\left(\tilde{W}_{(\bar{P}, \hat{Q}, \bar{Y}, \bar{C})}, \ln(\bar{P})\right)$, proving that $\lambda_{\mathcal{E}}^T = \bar{D}^T \left(I - \beta L\left(\tilde{W}_{(\bar{P}, \hat{Q}, \bar{Y}, \bar{C})}, \ln(\bar{P})\right)\right)^{-1}$. Since $\bar{D}^T = \xi^T$, the statement follows.

(iv). It is obvious. $\square$

Corollary 6. *Let T be generated by a profile of production functions G . If $x \in \mathbb{R}^n$ and T is Gateaux differentiable at $\Pi(x)$, then $\Sigma(\Pi(x)) = \{\tilde{W}\}$ and*

$$\Pi_0(x; d) = \Omega_{J_{\mathcal{M}(\tilde{W}, \cdot)}(\Pi(x))} d = \Omega_{J_T(\Pi(x))} d \quad \forall d \in \mathbb{R}^n. \quad (144)$$

Proof. Since T is Gateaux differentiable at $\Pi(x)$, so is T_i for all $i \in [n]$. By Corollary 4, we have that $\sigma_{T_i}(\Pi(x))$ is a singleton for all $i \in [n]$ and, in particular, $\Sigma(\Pi(x))$ is a singleton. This implies that $J_T(\Pi(x)) = J_{\mathcal{M}(\tilde{W}, \cdot)}(\Pi(x))$. By Theorem 2, the statement follows. $\square$

Remark 6. Fix $x \in \mathbb{R}^n$. Consider two stochastic matrices W and $\hat{W}$ and assume that $\Omega_{J_{\mathcal{M}(W, \cdot)}(x)} = \Omega_{J_{\mathcal{M}(\hat{W}, \cdot)}(x)}$. By the last part of Remark 5, it follows that $J_{\mathcal{M}(W, \cdot)}(x) = J_{\mathcal{M}(\hat{W}, \cdot)}(x)$. By definition of $\mathcal{M}$, it follows that $\nabla M(w_i, x) = \nabla M(\hat{w}_i, x)$ for all $i \in [n]$, that is,

$$\frac{e^{x_h} w_{ih}}{\sum_{j=1}^n e^{x_j} w_{ij}} = \frac{e^{x_h} \hat{w}_{ih}}{\sum_{j=1}^n e^{x_j} \hat{w}_{ij}} \quad \forall i, h \in [n].$$

For each $i \in [n]$ set $\lambda_i = \sum_{j=1}^n e^{x_j} w_{ij} / \sum_{j=1}^n e^{x_j} \hat{w}_{ij}$. We obtain that $w_{ih} = \lambda_i \hat{w}_{ih}$ for all $i, h \in [n]$ and, in particular, $1 = \sum_{h=1}^n w_{ih} = \lambda_i \sum_{h=1}^n \hat{w}_{ih} = \lambda_i$ for all $i \in [n]$. We can conclude that $w_{ih} = \hat{w}_{ih}$ for all $i, h \in [n]$, that is, $W = \hat{W}$. $\triangle$

Proof of Proposition 6. (i) is equivalent to (iii). By Corollary 4, the statement follows.

(iv) implies (iii). By point (ii) of Proposition 8, if $\mathfrak{E}$ is a singleton, then $\Sigma(\Pi(\bar{x})) = \{\tilde{W}_{\mathcal{E}}\}_{\mathcal{E} \in \mathfrak{E}}$ is a singleton.

(iii) implies (iv). By point (ii) of Proposition 8, if $\Sigma(\Pi(\bar{x}))$ is a singleton, so is $\{\tilde{W}_{\mathcal{E}}\}_{\mathcal{E} \in \mathfrak{E}}$. Assume there exist two equilibria $(\tilde{P}, \tilde{Q}, \tilde{Y}, \tilde{C}), (\bar{P}, \hat{Q}, \bar{Y}, \bar{C}) \in \mathfrak{E}$ that generate the unique $\tilde{W}_{\mathcal{E}}$. It follows that

$$\tilde{w}_{ij} = \frac{\hat{Q}_{ij}}{\sum_{k=1}^n \hat{Q}_{ik}} = \frac{\tilde{Q}_{ij}}{\sum_{k=1}^n \tilde{Q}_{ik}} \quad \forall i, j \in [n]. \quad (145)$$

For each $i \in [n]$ define $\mu_i = \sum_{k=1}^n \tilde{Q}_{ik} / \sum_{k=1}^n \hat{Q}_{ik}$. By Equation 145, we have that $(\tilde{Q}_{i1}, \dots, \tilde{Q}_{in}) = \mu_i (\hat{Q}_{i1}, \dots, \hat{Q}_{in})$ for all $i \in [n]$. By point (iv) of Proposition 8, we have that $\bar{P} = \tilde{P}$. By point (i) of Lemma 8 and since $(\hat{Q}_{i0}, \hat{Q}_i), (\tilde{Q}_{i0}, \tilde{Q}_i) \in \mathbb{R}_+^{n+1}$ solve (130) for all $i \in [n]$ with $P = \bar{P} = \tilde{P}$, then $\tilde{Q}_{i0} = \hat{Q}_{i0} = \frac{1-\beta}{Z_i} K_i(P)^\beta$ for all $i \in [n]$ and, for all $i \in [n]$:

$$\hat{Q}_{i0} + \sum_{j=1}^n \hat{Q}_{ij} \bar{P}_j = \tilde{Q}_{i0} + \sum_{j=1}^n \tilde{Q}_{ij} \bar{P}_j \implies \sum_{j=1}^n \hat{Q}_{ij} \bar{P}_j = \sum_{j=1}^n \tilde{Q}_{ij} \bar{P}_j = \mu_i \sum_{j=1}^n \hat{Q}_{ij} \bar{P}_j \implies \mu_i = 1. \quad (146)$$

This implies that $\hat{Q}_{ij} = \tilde{Q}_{ij}$ for all $i, j \in [n]$, proving that $\hat{Q} = \tilde{Q}$. By the proof of existence of a competitive equilibrium (cf. Proposition 8), we have that $\tilde{Y} = \hat{Y}$ is uniquely pinned down by $\bar{P} = \tilde{P}$ and $\hat{Q} = \tilde{Q}$. Given the preferences of the agent and $\bar{P} = \tilde{P}$, we have that $\bar{C} = \tilde{C}$, proving $(\tilde{P}, \tilde{Q}, \tilde{Y}, \tilde{C}) = (\bar{P}, \hat{Q}, \bar{Y}, \bar{C})$ and uniqueness of the equilibrium.

(i) implies (ii). By Corollary 6, the implication follows.

(ii) implies (i). By Theorem 2 and as Π is Gateaux differentiable at $\bar{x}$, there exists $W \in \mathcal{W}$ such that

$$\Omega_{J_{\mathcal{M}(\tilde{W}, \cdot)}(\Pi(\bar{x}))} d \geq \Pi_0(\bar{x}; d) = Wd \quad \forall d \in \mathbb{R}^n, \forall \tilde{W} \in \Sigma(\Pi(\bar{x})). \quad (147)$$

It follows that $\Omega_{J_{\mathcal{M}(\tilde{W}, \cdot)}(\Pi(\bar{x}))} = W$ for all $\tilde{W} \in \Sigma(\Pi(\bar{x}))$. By Remark 6, we can conclude that $J_{\mathcal{M}(\tilde{W}, \cdot)}(\Pi(\bar{x})) = J_{\mathcal{M}(\hat{W}, \cdot)}(\Pi(\bar{x}))$, that is $\tilde{W} = \hat{W}$, for all $\tilde{W}, \hat{W} \in \Sigma(\Pi(\bar{x}))$, proving that $\Sigma(\Pi(\bar{x}))$ is a singleton and that T is Gateaux differentiable at $\Pi(\bar{x})$.

The final conclusion thus follows by Rademacher's theorem as Π is Lipschitz continuous. $\square$

Corollary 7. *Let T be a log-cost operator and $\beta \in (0, 1)$. If $x \in \mathbb{R}^n$, then for each $d \in \mathbb{R}^n$ there exists $\tilde{W}_d \in \Sigma(\Pi(x))$ such that*

$$\lim_{h \downarrow 0} \frac{\Upsilon(x + hd) - \Upsilon(x)}{h} = -\xi^T \Omega_{J_{\mathcal{M}(\tilde{W}_d, \cdot)}(\Pi(x))} d \geq -\xi^T \Omega_{J_{\mathcal{M}(\tilde{W}, \cdot)}(\Pi(x))} d \quad \forall \tilde{W} \in \Sigma(\Pi(x)).$$

In particular, we have that

$$\lim_{h \downarrow 0} \frac{\Upsilon(x + hd) - \Upsilon(x)}{h} = \max_{\tilde{W} \in \Sigma(\Pi(x))} -\xi^T \Omega_{J_{\mathcal{M}(\tilde{W}, \cdot)}(\Pi(x))} d.$$

Proof. Fix $x, d \in \mathbb{R}^n$. By Theorem 2, there exists $\tilde{W}_d \in \Sigma(\Pi(x))$ such that

$$\lim_{h \downarrow 0} \frac{\Pi(x + hd) - \Pi(x)}{h} = \Omega_{J_{\mathcal{M}(\tilde{W}_d, \cdot)}(\Pi(x))} d \leq \Omega_{J_{\mathcal{M}(\tilde{W}, \cdot)}(\Pi(x))} d \quad \forall \tilde{W} \in \Sigma(\Pi(x)).$$

By definition of Υ and since $\xi \geq 0$, this implies that

$$\begin{aligned} \lim_{h \downarrow 0} \frac{\Upsilon(x + hd) - \Upsilon(x)}{h} &= \sum_{i=1}^n -\xi_i \lim_{h \downarrow 0} \frac{\Pi_i(x + hd) - \Pi_i(x)}{h} = -\xi^T \lim_{h \downarrow 0} \frac{\Pi(x + hd) - \Pi(x)}{h} \\ &= -\xi^T \Omega_{J_{\mathcal{M}(\tilde{W}_d, \cdot)}(\Pi(x))} d \geq -\xi^T \Omega_{J_{\mathcal{M}(\tilde{W}, \cdot)}(\Pi(x))} d \quad \forall \tilde{W} \in \Sigma(\Pi(x)), \end{aligned}$$

proving the statement. $\square$

Proof of Proposition 7. Fix $d \in \mathbb{R}^n$. By Corollary 7, we have that

$$\lim_{h \downarrow 0} \frac{\Upsilon(\bar{x} + hd) - \Upsilon(\bar{x})}{h} = \max_{\tilde{W} \in \Sigma(\Pi(\bar{x}))} -\xi^T \Omega_{J_{\mathcal{M}(\tilde{W}, \cdot)}(\Pi(\bar{x}))} d.$$

By points (ii) and (iii) of Proposition 8, we have that

$$\lim_{h \downarrow 0} \frac{\Upsilon(\bar{x} + hd) - \Upsilon(\bar{x})}{h} = (1 - \beta) \max_{\mathcal{E} \in \mathfrak{E}} -\lambda_{\mathcal{E}}^T d,$$

proving the statement. $\square$

D Proofs of the Results in Section 7

D.1 Semiconvexity of Υ

In this section, we are going to consider first a log-cost operator T which is twice continuously differentiable and we are going to express the Hessian of Π as a function of the Hessian of T . By Corollary 8, recall that Π inherits the differentiability properties. Before we start we introduce some notation. We consider the vector space $M_n(\mathbb{R})^n$ which is the space of vectors whose n components are $n \times n$ square matrices. Given a stochastic matrix W and $v \in M_n(\mathbb{R})^n$, the juxtaposition Wv is the vector whose i -th component is $\sum_{j=1}^n w_{ij} v_j$ and is itself an $n \times n$ square matrix. Finally, given a twice continuously differentiable operator $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we define by $H_S(x)$ the vector in $M_n(\mathbb{R})^n$ whose i -th component is the Hessian of S_i at x , denoted by $H_{S_i}(x)$. We say that $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is fundamentally C^2 if $\mathcal{D}_S$ is open and $\mathcal{D}_S = \mathcal{H}_S$. Thus, the operator T considered in Section 7 is assumed to be fundamentally C^2 .

Lemma 10. *Let T be a log-cost operator and $\beta \in (0, 1)$. If T is twice continuously differentiable, then*

$$H_{\Pi}(x) = \beta \sum_{t=0}^{\infty} \beta^t J_T(\Pi(x))^t v \quad \forall x \in \mathbb{R}^n \quad (148)$$

where $v \in M_n(\mathbb{R})^n$ is such that $v_i = J_{\Pi}(x)^T H_{T_i}(\Pi(x)) J_{\Pi}(x)$ for all $i \in [n]$.

Proof. Fix $x \in \mathbb{R}^n$. By definition of Π and since Π is twice continuously differentiable, we have that

$$J_{\Pi}(x) = (1 - \beta) I + \beta J_T(\Pi(x)) J_{\Pi}(x).$$

Fix $i, j \in [n]$. It follows that

$$\frac{\partial \Pi_i}{\partial x_j}(x) = (1 - \beta) e_j^i + \beta \sum_{l=1}^n \frac{\partial T_i}{\partial x_l}(\Pi(x)) \frac{\partial \Pi_l}{\partial x_j}(x). \quad (149)$$

Fix $h \in [n]$. We define $f_{il} : \mathbb{R}^n \rightarrow \mathbb{R}$ by $f_{il}(x) = \frac{\partial T_i}{\partial x_l}(\Pi(x)) \frac{\partial \Pi_l}{\partial x_j}(x)$ for all $x \in \mathbb{R}^n$ and for all

$l \in [n]$. By the Chain Rule, we obtain that

$$\frac{\partial f_{il}}{\partial x_h}(x) = \sum_{m=1}^n \frac{\partial^2 T_i}{\partial x_l \partial x_m}(\Pi(x)) \frac{\partial \Pi_m}{\partial x_h}(x) \frac{\partial \Pi_l}{\partial x_j}(x) + \frac{\partial T_i}{\partial x_l}(\Pi(x)) \frac{\partial^2 \Pi_l}{\partial x_j \partial x_h}(x) \quad \forall l \in [n].$$

By (149), this implies that

$$\begin{aligned} \frac{\partial^2 \Pi_i}{\partial x_j \partial x_h}(x) &= \beta \sum_{l=1}^n \frac{\partial f_{il}}{\partial x_h}(x) \\ &= \beta \sum_{l=1}^n \sum_{m=1}^n \frac{\partial \Pi_l}{\partial x_j}(x) \frac{\partial^2 T_i}{\partial x_l \partial x_m}(\Pi(x)) \frac{\partial \Pi_m}{\partial x_h}(x) + \beta \sum_{l=1}^n \frac{\partial T_i}{\partial x_l}(\Pi(x)) \frac{\partial^2 \Pi_l}{\partial x_j \partial x_h}(x). \end{aligned}$$

Since i, j , and h were arbitrarily chosen, we have that

$$H_{\Pi_i}(x) = \beta J_{\Pi}(x)^T H_{T_i}(\Pi(x)) J_{\Pi}(x) + \beta \sum_{l=1}^n \frac{\partial T_i}{\partial x_l}(\Pi(x)) H_{\Pi_l}(x) \quad \forall i \in [n].$$

If we define $\tilde{v} \in M_n(\mathbb{R})^n$ by $\tilde{v}_i = \frac{\beta}{1-\beta} J_{\Pi}(x)^T H_{T_i}(\Pi(x)) J_{\Pi}(x)$ for all $i \in [n]$, then we have that

$$H_{\Pi}(x) = (1 - \beta) \tilde{v} + \beta J_T(\Pi(x)) H_{\Pi}(x),$$

yielding that

$$H_{\Pi}(x) = \beta \sum_{t=0}^{\infty} \beta^t J_T(\Pi(x))^t v$$

and proving the statement. $\square$

Lemma 11. *Let T be a log-cost operator, $\beta \in (0, 1)$, and $\xi \in \Delta$. If T is twice continuously differentiable, then the map $\Upsilon : \mathbb{R}^n \rightarrow \mathbb{R}$, defined by $\Upsilon(x) = -\langle \xi, \Pi(x) \rangle$ for all $x \in \mathbb{R}^n$, is λ -semiconvex where $\lambda = \frac{\beta}{1-\beta}$, that is, $\Upsilon + \frac{\lambda}{2} \|\cdot\|_2^2$ is a convex function.*

Proof. Fix $x \in \mathbb{R}^n$. By Lemma 10, we have that

$$H_{\Pi}(x) = \frac{\beta}{1-\beta} (1 - \beta) \sum_{t=0}^{\infty} \beta^t J_T(\Pi(x))^t v = \frac{\beta}{1-\beta} \Omega_{J_T(\Pi(x))} v \quad (150)$$

where $v \in M_n(\mathbb{R})^n$ is such that $v_i = J_{\Pi}(x)^T H_{T_i}(\Pi(x)) J_{\Pi}(x)$ for all $i \in [n]$. Since $J_T(\Pi(x))$ is a stochastic matrix, so is $\Omega_{J_T(\Pi(x))}$ and, in particular, $\bar{\xi}$ defined as $\bar{\xi}^T = \xi^T \Omega_{J_T(\Pi(x))}$ belongs to Δ . It follows that

$$H_{\Upsilon}(x) = -\frac{\beta}{1-\beta} \xi^T \Omega_{J_T(\Pi(x))} v = -\frac{\beta}{1-\beta} \bar{\xi}^T v = -\frac{\beta}{1-\beta} \sum_{i=1}^n \bar{\xi}_i J_{\Pi}(x)^T H_{T_i}(\Pi(x)) J_{\Pi}(x) \quad (151)$$

$$= -\frac{\beta}{1-\beta} J_{\Pi}(x)^T \left(\sum_{i=1}^n \bar{\xi}_i H_{T_i}(\Pi(x)) \right) J_{\Pi}(x). \quad (152)$$

By Lemma 22, we have that $\max_{h \in B_{\infty}} \langle H_{T_i}(\Pi(x)) h, h \rangle \leq 1$ for all $i \in [n]$. Since $\bar{\xi} \in \Delta$, this

implies that

$$\max_{h \in B_\infty} \left\langle \sum_{i=1}^n \bar{\xi}_i H_{T_i}(\Pi(x)) h, h \right\rangle \leq \sum_{i=1}^n \bar{\xi}_i \max_{h \in B_\infty} \langle H_{T_i}(\Pi(x)) h, h \rangle \leq 1.$$

Since $J_\Pi(x)$ is a stochastic matrix, we have that $\|J_\Pi(x) d\|_\infty \leq \|d\|_\infty$ for all $d \in \mathbb{R}^n$. To simplify notation, set $A = \sum_{i=1}^n \bar{\xi}_i H_{T_i}(\Pi(x))$, $B = J_\Pi(x)$, and $C = B^T A B$. This implies that $\|Bd\|_\infty \leq \|d\|_\infty$ for all $d \in \mathbb{R}^n$ and

$$\max_{d \in B_\infty} \langle Cd, d \rangle = \max_{d \in B_\infty} \langle ABd, Bd \rangle = \max_{h \in \mathbb{R}^n: h=Bd \text{ and } d \in B_\infty} \langle Ah, h \rangle \leq \max_{h \in B_\infty} \langle Ah, h \rangle \leq 1.$$

Next, consider $d \in \mathbb{R}^n \setminus \{0\}$. If we set $\bar{d} = d / \|d\|_2$, then $\|\bar{d}\|_\infty \leq \|\bar{d}\|_2 = 1$ and, in particular, $\langle C\bar{d}, \bar{d} \rangle \leq 1$, that is, $\langle Cd, d \rangle \leq \|d\|_2^2$. Since d was arbitrarily chosen, we can conclude that $\langle Cd, d \rangle \leq \|d\|_2^2$ for all $d \in \mathbb{R}^n$. Since $H_\Upsilon(x) = -\frac{\beta}{1-\beta} J_\Pi(x)^T (\sum_{i=1}^n \bar{\xi}_i H_{T_i}(\Pi(x))) J_\Pi(x) = -\frac{\beta}{1-\beta} B^T A B = -\frac{\beta}{1-\beta} C$, we can conclude that

$$\langle -H_\Upsilon(x) d, d \rangle = \left\langle \frac{\beta}{1-\beta} Cd, d \right\rangle \leq \frac{\beta}{1-\beta} \|d\|_2^2 = \langle (\lambda I) d, d \rangle \quad \forall d \in \mathbb{R}^n,$$

proving that $\lambda I + H_\Upsilon(x) \succeq 0$ where $\lambda = \frac{\beta}{1-\beta}$. Since x was arbitrarily chosen and Υ is twice continuously differentiable, the statement follows. $\square$

Proposition 9. *If T is a log-cost operator, then there exists a sequence $\{T_m\}_{m \in \mathbb{N}}$ of log-cost operators such that $T_m \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ for all $m \in \mathbb{N}$ and*

$$\lim_m T_m(x) = T(x) \text{ and } \lim_m \Pi_m(x) = \Pi(x) \quad \forall x \in \mathbb{R}^n.$$

Proof. By Proposition 13, we have that for each $i \in [n]$ there exists a sequence $\{K_{i,m}\}_{m \in \mathbb{N}} \subseteq \mathcal{K}_{\text{nor}}$ such that $\{\kappa_{i,m}\}_{m \in \mathbb{N}} \subseteq C^\infty(\mathbb{R}^n)$ and $\kappa_{i,m} \xrightarrow{d} T_i$ where d is the distance on $C(\mathbb{R}^n)$ metrizing convergence over compacta. Define $T_m : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $T_{m,i}(x) = \kappa_{i,m}(x)$ for all $x \in \mathbb{R}^n$. It follows that each T_m is a production operator and belongs to $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Moreover, $\{T_m\}_{m \in \mathbb{N}}$ converges uniformly over compacta to T and, in particular, $\lim_m T_m(x) = T(x)$ for all $x \in \mathbb{R}^n$. Fix $x \in \mathbb{R}^n$. Since each T_m and Π are normalized and nonexpansive, we have that

$$\begin{aligned} \|\Pi_m(x) - \Pi(x)\|_\infty &= \|\beta T_m(\Pi_m(x)) - \beta T(\Pi(x))\|_\infty \\ &\leq \|\beta T_m(\Pi_m(x)) - \beta T_m(\Pi(x))\|_\infty + \|\beta T_m(\Pi(x)) - \beta T(\Pi(x))\|_\infty \\ &\leq \beta \|\Pi_m(x) - \Pi(x)\|_\infty + \beta \max_{z \in \mathbb{R}^n: \|z\|_\infty \leq \|x\|_\infty} \|T_m(z) - T(z)\|_\infty, \end{aligned}$$

proving that

$$\lim_m \|\Pi_m(x) - \Pi(x)\|_\infty \leq \frac{\beta}{1-\beta} \lim_m \max_{z \in \mathbb{R}^n: \|z\|_\infty \leq \|x\|_\infty} \|T_m(z) - T(z)\|_\infty = 0.$$

Since x was arbitrarily chosen, the statement follows. $\square$

Proposition 10. *If T is a log-cost operator, $\beta \in (0, 1)$, and $\xi \in \Delta$, then $\Upsilon : \mathbb{R}^n \rightarrow \mathbb{R}$, defined by $\Upsilon(x) = -\langle \xi, \Pi(x) \rangle$ for all $x \in \mathbb{R}^n$, is λ -semiconvex where $\lambda = \frac{\beta}{1-\beta}$, that is, $\Upsilon + \frac{\lambda}{2} \|\cdot\|_2^2$ is a convex function. In particular, we have that:*

1. Π_i is λ -semiconcave for all $i \in [n]$.
2. For each $x, d \in \mathbb{R}^n$, the function $t \mapsto \Upsilon_\circ(x + td; d)$ is locally of bounded variation.

Proof. By Proposition 9, there exists a sequence $\{T_m\}_{m \in \mathbb{N}} \subseteq C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ of log-cost operators such that $\lim_m \Pi_m(x) = \Pi(x)$ for all $x \in \mathbb{R}^n$. Define $\Upsilon_m : \mathbb{R}^n \rightarrow \mathbb{R}$ by $\Upsilon_m(x) = -\langle \xi, \Pi_m(x) \rangle$ for all $x \in \mathbb{R}^n$ and for all $m \in \mathbb{N}$. By Lemma 11, we have that $\Upsilon_m + \frac{\lambda}{2} \|\cdot\|_2^2$ is convex for all $m \in \mathbb{N}$ and $\lim_m \Upsilon_m(x) = \Upsilon(x)$ for all $x \in \mathbb{R}^n$. Since convexity passes to the limit, we can conclude that $\Upsilon + \frac{\lambda}{2} \|\cdot\|_2^2$ is convex.

1. Fix $i \in [n]$ and set $\xi = e^i$. It follows that $\Upsilon = -\Pi_i$, proving the statement.

2. Fix $x, d \in \mathbb{R}^n$. Since $\Upsilon + \frac{\lambda}{2} \|\cdot\|_2^2$ is convex, we have that the function $t \mapsto \Upsilon_\circ(x + td; d) = [\Upsilon_\circ(x + td; d) + \lambda \langle x + td, d \rangle] - \lambda \langle x + td, d \rangle$, that is, it is the difference of two increasing functions, proving the statement. $\square$

Corollary 8. *If T is a log-cost operator, then $\Pi^{-1} = \frac{1}{1-\beta}(I - \beta T)$, that is, $\Pi^{-1}(p) = \frac{1}{1-\beta}(p - \beta T(p))$ for all $p \in \mathbb{R}^n$. In particular, if T is twice continuously differentiable in a neighborhood U , so is Π in the neighborhood $V = \Pi^{-1}(U)$.*

Proof. Fix $p \in \mathbb{R}^n$. Since Π is bijective by Theorem 1, there exists $x \in \mathbb{R}^n$ such that $\Pi(x) = p$, that is, $\Pi^{-1}(p) = x$. This implies that

$$\begin{aligned} p &= \Pi(\Pi^{-1}(p)) = \Pi(x) = (1 - \beta)x + \beta T(\Pi(x)) \\ &= (1 - \beta)\Pi^{-1}(p) + \beta T(\Pi(\Pi^{-1}(p))) = (1 - \beta)\Pi^{-1}(p) + \beta T(p), \end{aligned}$$

proving the statement. Since $\Pi^{-1} = \frac{1}{1-\beta}(I - \beta T)$ and T is twice continuously differentiable on U , so is Π^{-1} . Since T is a log-cost operator, we have that $J_T(p)$ is a stochastic matrix and, in particular, $J_{\Pi^{-1}}(p) = \frac{1}{1-\beta}(I - \beta J_T(p))$ is invertible for all $p \in U$. By the Inverse Function Theorem, we can conclude that Π is twice continuously differentiable on $V = \Pi^{-1}(U)$. $\square$

Lemma 12. *Let T be a log-cost operator and $\beta \in (0, 1)$. If T is fundamentally C^2 and $d = e^j$ for some $j \in [n]$, then Π is fundamentally C^2 , $\{t \in \mathbb{R} : x + td \in \mathcal{D}_\Pi\}$ is open for all $x \in \text{span}\{d\}^\perp$, and $\{t \in \mathbb{R} : x + td \in \mathcal{D}_\Pi\}^c$ is countable for almost all $x \in \text{span}\{d\}^\perp$.²⁰*

Proof. By Proposition 6, $\mathcal{D}_\Pi = \Pi^{-1}(\mathcal{D}_T)$. Since by Theorem 1 Π is a homeomorphism and since $\mathcal{D}_T$ is open, $\mathcal{D}_\Pi$ is open. By Corollary 8, Π is twice continuously differentiable on $\mathcal{D}_\Pi$. Therefore, $\mathcal{D}_\Pi \subseteq \mathcal{H}_\Pi \subseteq \mathcal{D}_\Pi$, so $\mathcal{H}_\Pi = \mathcal{D}_\Pi$ and Π is fundamentally C^2 . By Proposition 10, we have that Π_i is λ -semiconcave for all $i \in [n]$ where $\lambda = \frac{\beta}{1-\beta}$. By (Cannarsa and Sinestrari, 2004, Propositions 1.1.3 and 4.1.3), this implies that $\mathcal{D}_{\Pi_i}^c$ is a countably H^{n-1} -rectifiable set for all $i \in [n]$. By (Cannarsa and Sinestrari, 2004, Definition A.3.4), we have that $\mathcal{D}_\Pi^c = \cup_{i=1}^n \mathcal{D}_{\Pi_i}^c$ is a countably H^{n-1} -rectifiable set. By (Evans, 2025, Definition 1.9 and Theorem 2.1), it follows that $\mathcal{D}_\Pi^c \subseteq \mathcal{N} \cup (\cup_{k \in \mathbb{N}} \mathcal{N}_k)$ where $\mathcal{N}$ is Borel measurable with $H^{n-1}(\mathcal{N}) = 0$ and $\mathcal{N}_k =$

²⁰The set $\text{span}\{d\}^\perp$ is isomorphic to $\mathbb{R}^{n-1}$. Thus, “almost all” is with respect to the Lebesgue measure over $\mathbb{R}^{n-1}$.

$f_k(\mathbb{R}^{n-1}) \subseteq \mathbb{R}^n$ and $f_k : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^n$ is Lipschitz continuous for all $k \in \mathbb{N}$. For each $k, m \in \mathbb{N}$ define $\mathcal{N}_{k,m} = f_k(\mathbb{R}^{n-1} \cap \{z \in \mathbb{R}^{n-1} : -m \leq z_i \leq m \ \forall i \in [n-1]\})$ and $\bar{\mathcal{N}}_{k,m} = \mathcal{N} \cup \mathcal{N}_{k,m}$. By (Evans, 2025, Theorems 2.5 and 2.8) and since $H^{n-1}(\mathcal{N}) = 0$, we have that $H^{n-1}(\mathcal{N}_{k,m}) < \infty$ and $H^{n-1}(\bar{\mathcal{N}}_{k,m}) < \infty$ for all $k, m \in \mathbb{N}$. Fix $k, m \in \mathbb{N}$, $j \in [n]$, and $d = e^j$. Note that $\bar{\mathcal{N}}_{k,m}$ is a Borel set. Let $P : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ be the canonical projection deleting the j -th coordinate. Define $z \mapsto \text{card}(\bar{\mathcal{N}}_{k,m} \cap P^{-1}(z))$ on $\mathbb{R}^{n-1}$. By (Evans, 2025, Lemma 5.8), we have that

$$\int_{\mathbb{R}^{n-1}} \text{card}(\bar{\mathcal{N}}_{k,m} \cap P^{-1}(z)) dz \leq H^{n-1}(\bar{\mathcal{N}}_{k,m}) < \infty.$$

Since k and m were arbitrarily chosen, it follows that $\{z \in \mathbb{R}^{n-1} : \text{card}(\bar{\mathcal{N}}_{k,m} \cap P^{-1}(z)) = \infty\}$ has measure 0. Define $\mathcal{Z} = \cap_{k,m \in \mathbb{N}} \{z \in \mathbb{R}^{n-1} : \text{card}(\bar{\mathcal{N}}_{k,m} \cap P^{-1}(z)) < \infty\}$, that is, $\mathcal{Z}^c = \cup_{k,m \in \mathbb{N}} \{z \in \mathbb{R}^{n-1} : \text{card}(\bar{\mathcal{N}}_{k,m} \cap P^{-1}(z)) = \infty\}$. Note that $\mathcal{Z}^c$ has measure 0. Fix $z \in \mathcal{Z}$. Consider $x \in \mathbb{R}^n$ such that $P(x) = z$. We have that $\text{card}(\bar{\mathcal{N}}_{k,m} \cap P^{-1}(z)) < \infty$, that is, $\{t \in \mathbb{R} : x + te^j \in \bar{\mathcal{N}}_{k,m}\}$ is finite for all $k, m \in \mathbb{N}$ and, in particular, $\cup_{k \in \mathbb{N}} \cup_{m \in \mathbb{N}} \{t \in \mathbb{R} : x + te^j \in \bar{\mathcal{N}}_{k,m}\}$ is countable. Since

$$\begin{aligned} \cup_{k \in \mathbb{N}} \cup_{m \in \mathbb{N}} \{t \in \mathbb{R} : x + te^j \in \bar{\mathcal{N}}_{k,m}\} &= \cup_{k \in \mathbb{N}} \{t \in \mathbb{R} : x + te^j \in \mathcal{N} \cup \mathcal{N}_k\} \\ &\supseteq \{t \in \mathbb{R} : x + te^j \in \mathcal{D}_{\Pi}^c\}, \end{aligned}$$

the latter set is countable. $\square$

D.2 Infinitesimal Changes in Variance

Consider a probability density function $\varphi : \mathbb{R} \rightarrow [0, \infty)$, that is, $\varphi \geq 0$ and $\int_{\mathbb{R}} \varphi(t) dt = 1$. We assume that φ is continuous, even, and such that $\int_{\mathbb{R}} t^2 \varphi(t) dt = 1$. For each $(\mu, \varepsilon) \in \mathbb{R} \times (0, \infty)$ define $\varphi_{\mu, \varepsilon} : \mathbb{R} \rightarrow [0, \infty)$ by $\varphi_{\mu, \varepsilon}(t) = \varepsilon^{-1} \varphi\left(\frac{t-\mu}{\varepsilon}\right)$ for all $t \in \mathbb{R}$. By the Change of Variable Formula (see, e.g., (Bogachev, 2007, Theorem 3.7.1)), note that $\varphi_{\mu, \varepsilon}$ is a probability density function with mean μ and variance ε^2 . Consider an L -Lipschitz function $f : \mathbb{R} \rightarrow \mathbb{R}$. By the Change of Variable Formula, define $u : \mathbb{R} \times (0, \infty) \rightarrow \mathbb{R}$ by

$$u(\mu, \varepsilon) = \int_{\mathbb{R}} f(t) \varphi_{\mu, \varepsilon}(t) dt = \int_{\mathbb{R}} f(\mu + \varepsilon s) \varphi(s) ds \quad \forall (\mu, \varepsilon) \in \mathbb{R} \times (0, \infty). \quad (153)$$

Since f is L -Lipschitz and $\int_{\mathbb{R}} |t| \varphi_{\mu, \varepsilon}(t) dt < \infty$, we have that $|f(t)| \leq L|t| + |f(0)|$ for all $t \in \mathbb{R}$ and, in particular, u is well defined and $u(\mu, \varepsilon)$ is the expected value of f when the probability density function has shape φ , mean μ , and standard deviation ε . Before discussing the main result of this section, we need to discuss some integration concepts and some notation. In what follows, we will have an integrator function $\gamma : \mathbb{R} \rightarrow \mathbb{R}$ which is right-continuous and such that $\gamma = \gamma_1 - \gamma_2$ where $\gamma_1, \gamma_2 : \mathbb{R} \rightarrow \mathbb{R}$ are increasing and right-continuous.²¹ In what follows, we will fix a decomposition (γ_1, γ_2) of γ . By (Billingsley, 1995, Theorem 12.4), note that there exists a unique σ -finite measure $\mu_{\gamma_i} : \mathcal{B}(\mathbb{R}) \rightarrow [0, \infty]$ such that $\mu_{\gamma_i}((a, b]) = \gamma_i(b) - \gamma_i(a)$ for all $a, b \in \mathbb{R}$ such that $b > a$ and for all $i \in \{1, 2\}$. Consider a measurable function h which is bounded on a bounded interval and 0 otherwise. We have that $\int_{\mathbb{R}} h d\mu_{\gamma_i}$ is well defined and

²¹Note that this is equivalent to assuming that γ is locally of bounded variation (see e.g., Propositions 13.1.1 and 13.1.3 in Simonnet, 1996).

finite for all $i \in \{1, 2\}$. We define $\int_{\mathbb{R}} h(t) d\gamma_i(t) \stackrel{\text{def}}{=} \int_{\mathbb{R}} h d\mu_{\gamma_i}$ for all $i \in \{1, 2\}$ and we set

$$\int_{\mathbb{R}} h(t) d\gamma(t) \stackrel{\text{def}}{=} \int_{\mathbb{R}} h(t) d\gamma_1(t) - \int_{\mathbb{R}} h(t) d\gamma_2(t)$$

Since γ_1 and γ_2 need not be bounded, it is important to observe that, a priori, we cannot define μ_{γ} as the difference of μ_{γ_1} and μ_{γ_2} : there might exist $A \in \mathcal{B}(\mathbb{R})$ such that $\mu_{\gamma_1}(A) = \mu_{\gamma_2}(A) = \infty$. Next, consider a function $\Xi : \mathbb{R} \rightarrow \mathbb{R}$ which is continuously differentiable and increasing and, in particular, Ξ' is measurable and nonnegative. Define $\tilde{\mu}_{\Xi}(A) = \int_A \Xi'(t) dt$ for all $A \in \mathcal{B}(\mathbb{R})$. By the Fundamental Theorem of Calculus and since the Riemann and the Lebesgue integrals coincide over closed and bounded intervals for continuous integrands (see, e.g., (Billingsley, 1995, p. 222)), we have that $\tilde{\mu}_{\Xi}((a, b]) = \Xi(b) - \Xi(a)$ for all $a, b \in \mathbb{R}$ such that $b > a$. By (Billingsley, 1995, Theorems 12.4 and 16.11), we can conclude that $\mu_{\Xi} = \tilde{\mu}_{\Xi}$ and $\int_{\mathbb{R}} h(t) d\Xi(t) = \int_{\mathbb{R}} h(t) \Xi'(t) dt$. Finally, given an integrator γ as above, we define $\Delta\gamma : \mathbb{R} \rightarrow \mathbb{R}$ by $\Delta\gamma(t) = \gamma(t) - \lim_k \gamma(t - 1/k)$ for all $t \in \mathbb{R}$. If γ is increasing, then $\Delta\gamma \geq 0$. Finally, observe that γ is differentiable almost everywhere. We denote by γ' its derivative which we set to be equal to 0 outside the domain of differentiability.

Lemma 13. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be an L -Lipschitz function such that $f'_+, f'_- : \mathbb{R} \rightarrow \mathbb{R}$ are well defined functions such that $\{t \in \mathbb{R} : f'_+(t) \neq f'_-(t)\}$ is a countable set. If $\gamma : \mathbb{R} \rightarrow \mathbb{R}$ is right-continuous, locally of bounded variation, and such that $\{t \in \mathbb{R} : \gamma(t) \neq f'_+(t)\}$ is a countable set, then*

$$\frac{\partial u}{\partial \varepsilon}(\mu, \varepsilon) = \int_{\mathbb{R}} s\gamma(\mu + \varepsilon s) \varphi(s) ds \in \mathbb{R} \quad \forall (\mu, \varepsilon) \in \mathbb{R} \times (0, \infty) \quad (154)$$

and

$$\frac{\partial u}{\partial \varepsilon}(\mu, \varepsilon) = \lim_k \int_{(\mu - \varepsilon k, \mu + \varepsilon k]} \left(\int_{[\frac{t - \mu}{\varepsilon}, k]} s\varphi(s) ds \right) d\gamma(t) \quad \forall (\mu, \varepsilon) \in \mathbb{R} \times (0, \infty). \quad (155)$$

Moreover, we have that:

1. *If $t \mapsto \gamma(t) + \lambda t$ is increasing with $\lambda > 0$, continuously differentiable on an open set O such that $O^c = \{t_k\}_{k \in \mathbb{N}}$ is countable, and $M : [0, \infty) \rightarrow [0, \infty)$ is defined by $M(t) = \int_{[t, \infty)} s\varphi(s) ds$ for all $t \in [0, \infty)$, then $\Delta\gamma \geq 0$ and*

$$\frac{\partial u}{\partial \varepsilon}(\mu, \varepsilon) = \int_{\mathbb{R}} M\left(\left|\frac{t - \mu}{\varepsilon}\right|\right) \gamma'(t) dt + \sum_{k=1}^{\infty} M\left(\left|\frac{t_k - \mu}{\varepsilon}\right|\right) \Delta\gamma(t_k) \quad \forall (\mu, \varepsilon) \in \mathbb{R} \times (0, \infty).$$

2. *If γ is as in Point 1 and $\varphi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$ for all $t \in \mathbb{R}$, then*

$$\frac{\partial u}{\partial \varepsilon}(\mu, \varepsilon) = \varepsilon \int_{\mathbb{R}} \varphi_{\mu, \varepsilon}(t) \gamma'(t) dt + \varepsilon \sum_{k=1}^{\infty} \varphi_{\mu, \varepsilon}(t_k) \Delta\gamma(t_k) \quad \forall (\mu, \varepsilon) \in \mathbb{R} \times (0, \infty). \quad (156)$$

Proof. Fix $(\mu, \varepsilon) \in \mathbb{R} \times (0, \infty)$. Since f is L -Lipschitz, we have that

$$\left| \frac{f(\mu + (\varepsilon + \delta)s) - f(\mu + \varepsilon s)}{\delta} \right| \leq L|s| \quad \forall s \in \mathbb{R}, \forall \delta \in \mathbb{R} \setminus \{0\}.$$

Since f is continuous and f'_- and f'_+ are well defined, if we consider a sequence $\{\delta_k\}_{k \in \mathbb{N}} \subseteq (0, \infty)$ such that $\lim_k \delta_k = 0$, we have that

$$\lim_k \frac{f(\mu + (\varepsilon + \delta_k)s) - f(\mu + \varepsilon s)}{\delta_k} = \begin{cases} sf'_+(\mu + \varepsilon s) & s > 0 \\ 0 & s = 0 \\ sf'_-(\mu + \varepsilon s) & s < 0 \end{cases},$$

yielding that $s \mapsto \frac{f(\mu + (\varepsilon + \delta_k)s) - f(\mu + \varepsilon s)}{\delta_k}$ is measurable for all $k \in \mathbb{N}$ and so is the pointwise limit of this sequence. Moreover, since $\int_{\mathbb{R}} |t| \varphi(t) dt < \infty$, we have that $s \mapsto \left| \frac{f(\mu + (\varepsilon + \delta_k)s) - f(\mu + \varepsilon s)}{\delta_k} \right|$ is bounded by $s \mapsto L|s|$ for all $k \in \mathbb{N}$ and, in particular, it is integrable for all $k \in \mathbb{N}$ and so is the pointwise limit of this sequence. By Lebesgue's Dominated Convergence Theorem, this implies that if $\{\delta_k\}_{k \in \mathbb{N}} \subseteq (0, \infty)$ (resp. $\{\delta_k\}_{k \in \mathbb{N}} \subseteq (-\infty, 0)$), then

$$\begin{aligned} \lim_k \frac{u(\mu, \varepsilon + \delta_k) - u(\mu, \varepsilon)}{\delta_k} &= \lim_k \int_{\mathbb{R}} \left(\frac{f(\mu + (\varepsilon + \delta_k)s) - f(\mu + \varepsilon s)}{\delta_k} \right) \varphi(s) ds \\ &= \int_{\mathbb{R}} \lim_k \left(\frac{f(\mu + (\varepsilon + \delta_k)s) - f(\mu + \varepsilon s)}{\delta_k} \right) \varphi(s) ds \\ &= \int_{(-\infty, 0]} sf'_-(\mu + \varepsilon s) \varphi(s) ds + \int_{(0, \infty)} sf'_+(\mu + \varepsilon s) \varphi(s) ds \\ (\text{resp. } &= \int_{(-\infty, 0]} sf'_+(\mu + \varepsilon s) \varphi(s) ds + \int_{(0, \infty)} sf'_-(\mu + \varepsilon s) \varphi(s) ds). \end{aligned}$$

Since $\{t \in \mathbb{R} : f'_+(t) \neq f'_-(t)\}$ and $\{t \in \mathbb{R} : \gamma(t) \neq f'_+(t)\}$ are countable sets, we can conclude that

$$\begin{aligned} &\int_{(-\infty, 0]} sf'_-(\mu + \varepsilon s) \varphi(s) ds + \int_{(0, \infty)} sf'_+(\mu + \varepsilon s) \varphi(s) ds \\ &= \int_{(-\infty, 0]} sf'_+(\mu + \varepsilon s) \varphi(s) ds + \int_{(0, \infty)} sf'_-(\mu + \varepsilon s) \varphi(s) ds \\ &= \int_{\mathbb{R}} sf'_+(\mu + \varepsilon s) \varphi(s) ds = \int_{\mathbb{R}} s\gamma(\mu + \varepsilon s) \varphi(s) ds. \end{aligned}$$

Since $\{\delta_k\}_{k \in \mathbb{N}}$ was arbitrarily chosen, this implies that $u(\mu, \cdot) : (0, \infty) \rightarrow \mathbb{R}$ has left- and right-derivative at ε , the two side-derivatives coincide, and their value is $\int_{\mathbb{R}} s\gamma(\mu + \varepsilon s) \varphi(s) ds$. Since μ and ε were arbitrarily chosen, (154) follows. Fix again $(\mu, \varepsilon) \in \mathbb{R} \times (0, \infty)$. Define $\Phi : \mathbb{R} \rightarrow [0, \infty)$ by $\Phi(t) = \int_{[0, t]} s\varphi(s) ds$ for all $t \in [0, \infty)$ and $\Phi(t) = 0$ for all $t \in (-\infty, 0)$. Define $\Psi : \mathbb{R} \rightarrow (-\infty, 0]$ by $\Psi(t) = \int_{[t, 0]} s\varphi(s) ds$ for all $t \in (-\infty, 0]$ and $\Psi(t) = 0$ for all $t \in (0, \infty)$. By the Fundamental Theorem of Calculus and since the Riemann and the Lebesgue integral coincide over closed and bounded intervals for continuous integrands (see, e.g., Billingsley (1995)), note that Φ is continuously differentiable, increasing, and such that $\Phi(0) = 0$ and $\Phi'(t) = t\varphi(t)$ (resp. $= 0$) for all $t \in [0, \infty)$ (resp. $t \in (-\infty, 0)$). Note also that $\Psi(t) = -\Phi(-t)$ for all $t \in \mathbb{R}$ and, in particular, Ψ is continuously differentiable, increasing, and $\Psi(t) = 0$ for all $t > 0$ and $\Psi(t) = \int_t^0 s\varphi(s) ds$ for all $t \leq 0$. Define $\Phi_{\mu, \varepsilon} : \mathbb{R} \rightarrow [0, \infty)$ by $\Phi_{\mu, \varepsilon}(t) = \Phi\left(\frac{t - \mu}{\varepsilon}\right)$ for all $t \in \mathbb{R}$ and $\Psi_{\mu, \varepsilon} : \mathbb{R} \rightarrow (-\infty, 0]$ by $\Psi_{\mu, \varepsilon}(t) = \Psi\left(\frac{t - \mu}{\varepsilon}\right)$ for all $t \in \mathbb{R}$.

Given γ , consider a decomposition (γ_1, γ_2) . Fix $i \in \{1, 2\}$. By the Change of Variable Formula and Integration by Parts (see, e.g., (Billingsley, 1995, Theorem 18.4)) and since γ_i , $\Phi_{\mu, \varepsilon}$, and $\Psi_{\mu, \varepsilon}$ are right-continuous and increasing, we have for all $k \in \mathbb{N}$ that

$$\begin{aligned}
& \int_{(0, k]} s \gamma_i(\mu + \varepsilon s) \varphi(s) ds = \int_{(0, k]} \gamma_i(\mu + \varepsilon s) \Phi'(s) ds = \int_{(\mu, \mu + \varepsilon k]} \gamma_i(t) \frac{1}{\varepsilon} \Phi' \left(\frac{t - \mu}{\varepsilon} \right) dt \\
& = \int_{(\mu, \mu + \varepsilon k]} \gamma_i(t) \Phi'_{\mu, \varepsilon}(t) dt = \int_{(\mu, \mu + \varepsilon k]} \gamma_i(t) d\Phi_{\mu, \varepsilon}(t) \\
& = \Phi_{\mu, \varepsilon}(\mu + \varepsilon k) \gamma_i(\mu + \varepsilon k) - \Phi_{\mu, \varepsilon}(\mu) \gamma_i(\mu) - \int_{(\mu, \mu + \varepsilon k]} \Phi_{\mu, \varepsilon}(t) d\gamma_i(t) \\
& = \Phi_{\mu, \varepsilon}(\mu + \varepsilon k) \gamma_i(\mu + \varepsilon k) - \int_{(\mu, \mu + \varepsilon k]} \Phi_{\mu, \varepsilon}(t) d\gamma_i(t) \\
& = \Phi(k) \gamma_i(\mu + \varepsilon k) - \int_{(\mu, \mu + \varepsilon k]} \Phi \left(\left| \frac{t - \mu}{\varepsilon} \right| \right) d\gamma_i(t)
\end{aligned}$$

and

$$\begin{aligned}
& \int_{(-k, 0]} s \gamma_i(\mu + \varepsilon s) \varphi(s) ds = - \int_{(-k, 0]} \gamma_i(\mu + \varepsilon s) \Psi'(s) ds = - \int_{(\mu - \varepsilon k, \mu]} \gamma_i(t) \frac{1}{\varepsilon} \Psi' \left(\frac{t - \mu}{\varepsilon} \right) dt \\
& = - \int_{(\mu - \varepsilon k, \mu]} \gamma_i(t) \Psi'_{\mu, \varepsilon}(t) dt = - \int_{(\mu - \varepsilon k, \mu]} \gamma_i(t) d\Psi_{\mu, \varepsilon}(t) \\
& = - \left[\Psi_{\mu, \varepsilon}(\mu) \gamma_i(\mu) - \Psi_{\mu, \varepsilon}(\mu - \varepsilon k) \gamma_i(\mu - \varepsilon k) - \int_{(\mu - \varepsilon k, \mu]} \Psi_{\mu, \varepsilon}(t) d\gamma_i(t) \right] \\
& = \Psi_{\mu, \varepsilon}(\mu - \varepsilon k) \gamma_i(\mu - \varepsilon k) + \int_{(\mu - \varepsilon k, \mu]} \Psi_{\mu, \varepsilon}(t) d\gamma_i(t) \\
& = \Psi(-k) \gamma_i(\mu - \varepsilon k) + \int_{(\mu - \varepsilon k, \mu]} \Psi \left(\frac{t - \mu}{\varepsilon} \right) d\gamma_i(t) \\
& = -\Phi(k) \gamma_i(\mu - \varepsilon k) - \int_{(\mu - \varepsilon k, \mu]} \Phi \left(\left| \frac{t - \mu}{\varepsilon} \right| \right) d\gamma_i(t).
\end{aligned}$$

By adding these two terms, it follows for all $k \in \mathbb{N}$ that

$$\begin{aligned}
& \int_{(-k, k]} s \gamma_i(\mu + \varepsilon s) \varphi(s) ds \\
& = -\Phi(k) \gamma_i(\mu - \varepsilon k) - \int_{(\mu - \varepsilon k, \mu]} \Phi \left(\left| \frac{t - \mu}{\varepsilon} \right| \right) d\gamma_i(t) + \Phi(k) \gamma_i(\mu + \varepsilon k) - \int_{(\mu, \mu + \varepsilon k]} \Phi \left(\left| \frac{t - \mu}{\varepsilon} \right| \right) d\gamma_i(t) \\
& = \Phi(k) [\gamma_i(\mu + \varepsilon k) - \gamma_i(\mu - \varepsilon k)] - \int_{(\mu - \varepsilon k, \mu + \varepsilon k]} \Phi \left(\left| \frac{t - \mu}{\varepsilon} \right| \right) d\gamma_i(t) \\
& = \int_{(\mu - \varepsilon k, \mu + \varepsilon k]} \left[\Phi(k) - \Phi \left(\left| \frac{t - \mu}{\varepsilon} \right| \right) \right] d\gamma_i(t) \\
& = \int_{(\mu - \varepsilon k, \mu + \varepsilon k]} \left(\int_{\left[\left| \frac{t - \mu}{\varepsilon} \right|, k \right]} s \varphi(s) ds \right) d\gamma_i(t).
\end{aligned}$$

Since i was arbitrarily chosen and each integral is finite, we can conclude that

$$\int_{(-k,k]} s\gamma(\mu + \varepsilon s) \varphi(s) ds = \int_{(\mu - \varepsilon k, \mu + \varepsilon k]} \left(\int_{[\lfloor \frac{t-\mu}{\varepsilon} \rfloor, k]} s\varphi(s) ds \right) d\gamma(t)$$

By passing to the limit and applying Lebesgue's Dominated Convergence Theorem and since $\int_{\mathbb{R}} s f'_+(\mu + \varepsilon s) \varphi(s) ds \in \mathbb{R}$, we can conclude that

$$\int_{\mathbb{R}} s\gamma(\mu + \varepsilon s) \varphi(s) ds = \lim_k \int_{(\mu - \varepsilon k, \mu + \varepsilon k]} \left(\int_{[\lfloor \frac{t-\mu}{\varepsilon} \rfloor, k]} s\varphi(s) ds \right) d\gamma(t).$$

Since μ and ε were arbitrarily chosen, (155) follows.

1. Given γ , consider the decomposition (γ_1, γ_2) such that $\gamma_1(t) = \gamma(t) + \lambda t$ and $\gamma_2(t) = \lambda t$ for all $t \in \mathbb{R}$. Since γ_1 is increasing and continuous on O , we have that $\Delta\gamma_1$ is positive and such that $\Delta\gamma_1 = \Delta\gamma$ as well as $\Delta\gamma_1(t) = 0$ for all $t \in O$, proving that $\Delta\gamma$ is positive and such that $\Delta\gamma(t) = 0$ for all $t \in O$. By definition of μ_{γ_1} , we have that

$$\mu_{\gamma_1}(\{t\}) = \lim_k \mu_{\gamma_1}((t - 1/k, t]) = \lim_k [\gamma_1(t) - \gamma_1(t - 1/k)] = \Delta\gamma_1(t) = \Delta\gamma(t) \quad \forall t \in \mathbb{R}.$$

Since γ_1 is continuously differentiable on the open set O and O^c is countable, we have that γ'_1 is well defined and continuous on O and so is γ' . By convention, recall that we set $\gamma'_1(t) = 0$ for all $t \in O^c$. By Lebesgue's Decomposition Theorem (see, e.g., (Billingsley, 1995, Equation 32.8 and Exercise 32.7)), we have that $\mu_{\gamma_1} = \mu_{\gamma_1,ac} + \mu_{\gamma_1,s}$ where $\mu_{\gamma_1,ac}$ (resp. $\mu_{\gamma_1,s}$) is absolutely continuous (resp. singular) with respect to the Lebesgue measure. By the Radon-Nikodym Theorem (see, e.g., (Billingsley, 1995, Theorem 32.2)), there exists a measurable function $\tilde{\gamma} : \mathbb{R} \rightarrow [0, \infty)$ such that $\mu_{\gamma_1,ac}(A) = \int_A \tilde{\gamma}(t) dt$ for all $A \in \mathcal{B}(\mathbb{R})$. It follows that $\tilde{\gamma}$ can be replaced by γ'_1 and $\mu_{\gamma_1,ac}(O^c) = 0$ as well as $\mu_{\gamma_1,s}(O) = 0$. Since $\mu_{\gamma_1,s}$ is countably additive and $O^c = \{t_k\}_{k \in \mathbb{N}}$ is countable, it follows that

$$\begin{aligned} \mu_{\gamma_1}(A) &= \mu_{\gamma_1,ac}(A) + \mu_{\gamma_1,s}(A) = \mu_{\gamma_1,ac}(A) + \mu_{\gamma_1,s}(A \cap O^c) \\ &= \int_A \gamma'_1(t) dt + \sum_{k: t_k \in A} \mu_{\gamma_1,s}(\{t_k\}) \quad \forall A \in \mathcal{B}(\mathbb{R}). \end{aligned}$$

Since $0 \leq \mu_{\gamma_1,s} \leq \mu_{\gamma_1}$, note that $\mu_{\gamma_1,s}(A) = \lim_m \sum_{k=1}^m \Delta\gamma(t_k) \delta_{t_k}(A)$ for all $A \in \mathcal{B}(\mathbb{R})$, that is, $\mu_{\gamma_1,s} = \sum_{k=1}^{\infty} \Delta\gamma(t_k) \delta_{t_k}$ where δ_{t_k} is the Dirac measure at t_k for all $k \in \mathbb{N}$. Fix $k \in \mathbb{N}$. Define $M_k : \mathbb{R} \rightarrow [0, \infty)$ by $M_k(t) = \int_{[\lfloor \frac{t-\mu}{\varepsilon} \rfloor, k]} s\varphi(s) ds = \int 1_{[\lfloor \frac{t-\mu}{\varepsilon} \rfloor, k]}(s) s\varphi(s) ds$ for all $t \in \mathbb{R}$. We can conclude that

$$\begin{aligned} &\int_{(\mu - \varepsilon k, \mu + \varepsilon k]} \left(\int_{[\lfloor \frac{t-\mu}{\varepsilon} \rfloor, k]} s\varphi(s) ds \right) d\gamma_1(t) \\ &= \int_{\mathbb{R}} 1_{(\mu - \varepsilon k, \mu + \varepsilon k]} M_k d\mu_{\gamma_1} = \int_{\mathbb{R}} 1_{(\mu - \varepsilon k, \mu + \varepsilon k]} M_k d\mu_{\gamma_1,ac} + \int_{\mathbb{R}} 1_{(\mu - \varepsilon k, \mu + \varepsilon k]} M_k d\mu_{\gamma_1,s} \\ &= \int_{\mathbb{R}} 1_{(\mu - \varepsilon k, \mu + \varepsilon k]}(t) M_k(t) \gamma'_1(t) dt + \int_{\mathbb{R}} 1_{(\mu - \varepsilon k, \mu + \varepsilon k]} M_k d\mu_{\gamma_1,s} \end{aligned}$$

By Tonelli's Theorem and since φ is even and

$$\begin{aligned} & 1_{[\mu-\varepsilon k, \mu+\varepsilon k]}(t) 1_{[\lfloor \frac{t-\mu}{\varepsilon} \rfloor, k]}(s) s\varphi(s) \\ &= 1_{[\mu-\varepsilon k, \mu+\varepsilon k]}(t) 1_{[\mu-\varepsilon s, \mu+\varepsilon s]}(t) 1_{[0, k]}(s) s\varphi(s) \geq 0 \quad \forall (s, t) \in \mathbb{R} \times \mathbb{R}, \end{aligned}$$

we have that

$$\begin{aligned} & \int_{(\mu-\varepsilon k, \mu+\varepsilon k]} \left(\int_{[\lfloor \frac{t-\mu}{\varepsilon} \rfloor, k]} s\varphi(s) ds \right) d\gamma_2(t) = \lambda \int_{(\mu-\varepsilon k, \mu+\varepsilon k]} \left(\int_{[\lfloor \frac{t-\mu}{\varepsilon} \rfloor, k]} s\varphi(s) ds \right) dt \\ &= \lambda \int_{[\mu-\varepsilon k, \mu+\varepsilon k]} \left(\int_{[\lfloor \frac{t-\mu}{\varepsilon} \rfloor, k]} s\varphi(s) ds \right) dt = \lambda \int_{\mathbb{R}} 1_{[\mu-\varepsilon k, \mu+\varepsilon k]}(t) \left(\int_{\mathbb{R}} 1_{[\lfloor \frac{t-\mu}{\varepsilon} \rfloor, k]}(s) s\varphi(s) ds \right) dt \\ &= \lambda \int_{\mathbb{R}} \left(\int_{\mathbb{R}} 1_{[\mu-\varepsilon k, \mu+\varepsilon k]}(t) 1_{[\lfloor \frac{t-\mu}{\varepsilon} \rfloor, k]}(s) s\varphi(s) ds \right) dt \\ &= \lambda \int_{\mathbb{R}} \left(\int_{\mathbb{R}} 1_{[\mu-\varepsilon k, \mu+\varepsilon k]}(t) 1_{[\mu-\varepsilon s, \mu+\varepsilon s]}(t) 1_{[0, k]}(s) s\varphi(s) ds \right) dt \\ &= \lambda \int_{\mathbb{R}} \left(\int_{\mathbb{R}} 1_{[\mu-\varepsilon k, \mu+\varepsilon k]}(t) 1_{[\mu-\varepsilon s, \mu+\varepsilon s]}(t) 1_{[0, k]}(s) s\varphi(s) dt \right) ds \\ &= \lambda \int_{\mathbb{R}} \left[1_{[0, k]}(s) s\varphi(s) \left(\int_{\mathbb{R}} 1_{[\mu-\varepsilon k, \mu+\varepsilon k]}(t) 1_{[\mu-\varepsilon s, \mu+\varepsilon s]}(t) dt \right) \right] ds \\ &= \lambda \int_{\mathbb{R}} \left[1_{[0, k]}(s) s\varphi(s) \left(\int_{\mathbb{R}} 1_{[\max\{\mu-\varepsilon k, \mu-\varepsilon s\}, \min\{\mu+\varepsilon s, \mu+\varepsilon k\}]} dt \right) \right] ds \\ &= \lambda \int_{\mathbb{R}} 1_{[0, k]}(s) s\varphi(s) (2\varepsilon s) ds = 2\lambda\varepsilon \int_{[0, k]} s^2\varphi(s) ds = \lambda\varepsilon \int_{[-k, k]} s^2\varphi(s) ds \end{aligned}$$

By the Monotone Convergence Theorem and since k was arbitrarily chosen, $M_{k+1} \geq M_k \geq 0$ for all $k \in \mathbb{N}$, and $\int_{\mathbb{R}} s^2\varphi(s) ds = 1$, we obtain that $\lim_k M_k(t) = M(\lfloor \frac{t-\mu}{\varepsilon} \rfloor)$ for almost all $t \in \mathbb{R}$ and

$$\begin{aligned} & \lim_k \int_{(\mu-\varepsilon k, \mu+\varepsilon k]} \left(\int_{[\lfloor \frac{t-\mu}{\varepsilon} \rfloor, k]} s\varphi(s) ds \right) d\gamma_1(t) \\ &= \int_{\mathbb{R}} M\left(\left\lfloor \frac{t-\mu}{\varepsilon} \right\rfloor\right) \gamma_1'(t) dt + \int_{\mathbb{R}} M\left(\left\lfloor \frac{t-\mu}{\varepsilon} \right\rfloor\right) d\mu_{\gamma_1, s} \\ &= \int_{\mathbb{R}} M\left(\left\lfloor \frac{t-\mu}{\varepsilon} \right\rfloor\right) \gamma_1'(t) dt + \sum_{k=1}^{\infty} M\left(\left\lfloor \frac{t_k - \mu}{\varepsilon} \right\rfloor\right) \Delta\gamma(t_k) \end{aligned}$$

and

$$\lim_k \int_{(\mu-\varepsilon k, \mu+\varepsilon k]} \left[\int_{[\lfloor \frac{t-\mu}{\varepsilon} \rfloor, k]} s\varphi(s) ds \right] d\gamma_2(t) = \int_{\mathbb{R}} M\left(\left\lfloor \frac{t-\mu}{\varepsilon} \right\rfloor\right) \lambda dt \quad (157)$$

and

$$\lim_k \int_{(\mu-\varepsilon k, \mu+\varepsilon k]} \left[\int_{[\lfloor \frac{t-\mu}{\varepsilon} \rfloor, k]} s\varphi(s) ds \right] d\gamma_2(t) = \lim_k \lambda\varepsilon \int_{[-k, k]} s^2\varphi(s) ds = \lambda\varepsilon \in \mathbb{R}. \quad (158)$$

By (157) and (158) and since $\gamma'_1(t) = \gamma'(t) + \lambda$ for almost every $t \in \mathbb{R}$, it follows that

$$\lim_k \int_{(\mu-\varepsilon k, \mu+\varepsilon k]} \left(\int_{[\lfloor \frac{t-\mu}{\varepsilon} \rfloor, k]} s \varphi(s) ds \right) d\gamma(t) = \int_{\mathbb{R}} M\left(\left\lfloor \frac{t-\mu}{\varepsilon} \right\rfloor\right) \gamma'(t) dt + \sum_{k=1}^{\infty} M\left(\left\lfloor \frac{t_k-\mu}{\varepsilon} \right\rfloor\right) \Delta\gamma(t_k),$$

proving the point.

2. Observe that $\varphi'(s) = -s\varphi(s)$ for all $s \in \mathbb{R}$. By the Monotone Convergence Theorem and the Fundamental Theorem of Calculus and since the Riemann and the Lebesgue integral coincide over closed and bounded intervals for continuous integrands and $\varphi \geq 0$, it follows that

$$\begin{aligned} M(t) &= \lim_k \int_{[t, k]} s \varphi(s) ds = - \lim_k \int_{[t, k]} \varphi'(s) ds = - \lim_k \int_t^k \varphi'(s) ds \\ &= \lim_k [\varphi(t) - \varphi(k)] = \varphi(t) \quad \forall t \in [0, \infty). \end{aligned}$$

Since φ is even, it follows that $M(\lfloor \frac{t-\mu}{\varepsilon} \rfloor) = \varphi(\lfloor \frac{t-\mu}{\varepsilon} \rfloor) = \varphi(\frac{t-\mu}{\varepsilon}) = \varepsilon \varphi_{\mu, \varepsilon}(t)$ for all $t \in \mathbb{R}$, proving the statement. $\square$

Without loss of generality in what follows, we focus on the last coordinate of $\mathbb{R}^n$, that is, on the n -th sector. Consider a probability density function $\varphi : \mathbb{R} \rightarrow [0, \infty)$ which is continuous and even with standard deviation 1. For example, φ could be the normal distribution $\varphi(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$ for all $t \in \mathbb{R}$. If we fix $(\mu, \varepsilon) \in \mathbb{R} \times (0, \infty)$, then $\varphi_{\mu, \varepsilon} : \mathbb{R} \rightarrow [0, \infty)$, defined by $\varphi_{\mu, \varepsilon}(t) = \varepsilon^{-1} \varphi(\frac{t-\mu}{\varepsilon})$ for all $t \in \mathbb{R}$, is a probability density function with mean μ and standard deviation ε . Consider also a probability density $\tilde{\varphi} : \mathbb{R}^{n-1} \rightarrow [0, \infty)$ satisfying $\int_{\mathbb{R}^{n-1}} \|z\|_{\infty} \tilde{\varphi}(z) dz < \infty$. Since Υ is Lipschitz and both shock distributions have finite first moments, $\mathbb{E}\Upsilon : \mathbb{R} \times (0, \infty) \rightarrow \mathbb{R}$ given by

$$\mathbb{E}\Upsilon(\mu, \varepsilon) = \int_{\mathbb{R}} \dots \int_{\mathbb{R}} \Upsilon(x_1, \dots, x_{n-1}, x_n) \tilde{\varphi}(x_1, \dots, x_{n-1}) \varphi_{\mu, \varepsilon}(x_n) dx_1 \dots dx_{n-1} dx_n \quad \forall (\mu, \varepsilon) \in \mathbb{R} \times (0, \infty), \quad (159)$$

is well defined. In other words, $\mathbb{E}\Upsilon(\mu, \varepsilon)$ is the expected log-GDP when the first $n-1$ normalized log-productivities, $x_i = -\ln Z_i / (1-\beta)$, are jointly distributed according to $\tilde{\varphi}$ and x_n is independently distributed according to the rescaled φ , that is, with mean μ , standard deviation ε , and shape φ . The next result shows how infinitesimal changes in standard deviation ε affect this expectation. To this end, we need to introduce some extra notation. We define $\Upsilon_{\circ} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$\Upsilon_{\circ}(x; d) = \lim_{h \downarrow 0} \frac{\Upsilon(x + hd) - \Upsilon(x)}{h} \quad \forall (x, d) \in \mathbb{R}^n \times \mathbb{R}^n.$$

By Corollary 7, we have that $\Upsilon_{\circ}$ is well defined. By Proposition 7,

$$\Upsilon_{\circ}(x; e^n) = -(1-\beta)\lambda_n(x) \quad \forall x \in \mathbb{R}^n. \quad (160)$$

Define $V = \{x \in \mathbb{R}^n : x_n = 0\}$. By Proposition 10, for every $v \in V$, the function

$$t \mapsto \Upsilon_{\circ}(v + te^n; e^n) + \frac{\beta}{1-\beta} t \quad (161)$$

is finite, non-decreasing, and right-continuous. So, $t \mapsto \lambda_n(v + te^n)$ is right-continuous, locally of bounded variation, and has finite left limits. For every $v \in V$, its derivative exists for almost every $t \in \mathbb{R}$. By Tonelli's Theorem, the partial derivative $\partial \lambda_n / \partial x_n$ exists almost everywhere in $\mathbb{R}^n$.

Proposition 11. *Let T be a fundamentally C^2 log-cost operator, $\beta \in (0, 1)$, and $\xi \in \Delta$. Let $\tilde{\varphi}$ be a probability density on $\mathbb{R}^{n-1}$ satisfying $\int_{\mathbb{R}^{n-1}} \|z\|_\infty \tilde{\varphi}(z) dz < \infty$. If $\varphi : \mathbb{R} \rightarrow [0, \infty)$ is continuous, even, and such that $\int_{\mathbb{R}} \varphi(t) dt = 1$ and $\int_{\mathbb{R}} t^2 \varphi(t) dt = 1$, then for each $(\mu, \varepsilon) \in \mathbb{R} \times (0, \infty)$*

$$\begin{aligned} \frac{\partial \mathbb{E} \Upsilon}{\partial \varepsilon}(\mu, \varepsilon) = & \\ & - (1 - \beta) \int_{\mathbb{R}} \dots \int_{\mathbb{R}} M \left(\left| \frac{x_n - \mu}{\varepsilon} \right| \right) \frac{\partial \lambda_n}{\partial x_n}(x_1, \dots, x_n) \tilde{\varphi}(x_1, \dots, x_{n-1}) dx_1 \dots dx_{n-1} dx_n \\ & - (1 - \beta) \int_{\mathbb{R}} \dots \int_{\mathbb{R}} \left(\sum_{x_n \in \mathbb{R}} M \left(\left| \frac{x_n - \mu}{\varepsilon} \right| \right) \Delta \lambda_n(x_1, \dots, x_n) \right) \tilde{\varphi}(x_1, \dots, x_{n-1}) dx_1 \dots dx_{n-1} \end{aligned}$$

where $M : [0, \infty) \rightarrow [0, \infty)$ is defined by $M(t) = \int_{[t, \infty)} s \varphi(s) ds$ for all $t \in [0, \infty)$ and $\Delta \lambda_n(x) \leq 0$ for all $x \in \mathbb{R}^n$.

Proof. Given a vector $z \in \mathbb{R}^{n-1}$, we define $\tilde{z} \in \mathbb{R}^n$ to be such that $\tilde{z}_i = z_i$ for all $i \in [n-1]$ and $\tilde{z}_n = 0$, that is, $\tilde{z} \in V$. Define $\hat{u} : \mathbb{R}^{n-1} \times \mathbb{R} \times (0, \infty) \rightarrow \mathbb{R}$ by

$$\hat{u}(z, \mu, \varepsilon) = \int_{\mathbb{R}} \Upsilon(\tilde{z} + te^n) \varphi_{\mu, \varepsilon}(t) dt \quad \forall (z, \mu, \varepsilon) \in \mathbb{R}^{n-1} \times \mathbb{R} \times (0, \infty).$$

Define by $\mathcal{Z}$ the set of points z in $\mathbb{R}^{n-1}$ such that $O = \{t \in \mathbb{R} : \tilde{z} + te^n \in \mathcal{D}_\Pi\}$ is open and $\{t \in \mathbb{R} : \tilde{z} + te^n \in \mathcal{D}_\Pi\}^c$ is countable. By Lemma 12 and since T is fundamentally C^2 , observe that $\mathcal{Z}^c$ has Lebesgue measure 0 in $\mathbb{R}^{n-1}$. Fix $z \in \mathcal{Z}$. Define $f_z : \mathbb{R} \rightarrow \mathbb{R}$ by $f_z(t) = \Upsilon(\tilde{z} + te^n)$ for all $t \in \mathbb{R}$. Since Υ is Lipschitz continuous of order 1, so is f_z . By Proposition 6 and Corollary 7 and since $\{t \in \mathbb{R} : \tilde{z} + te^n \in \mathcal{D}_\Pi\}^c$ is countable, we have that f_z has left- and right-derivative at each point of $\mathbb{R}$ with

$$f'_{z,+}(t) = \Upsilon_\circ(\tilde{z} + te^n, e^n) \text{ and } f'_{z,-}(t) = -\Upsilon_\circ(\tilde{z} + te^n, -e^n) \quad \forall t \in \mathbb{R}.$$

Define $g_z(t) = f_z(t) + \frac{\beta}{2(1-\beta)} t^2$, $\gamma_z(t) := f'_{z,+}(t)$. The function g_z is convex, and $\gamma_z(t) = g'_{z,+}(t) - \frac{\beta}{1-\beta} t = -(1-\beta) \lambda_n(\tilde{z} + te^n)$. Consequently, γ_z is right-continuous and locally of bounded variation, and $t \mapsto \gamma_z(t) + \frac{\beta}{1-\beta} t$ is non-decreasing.

Since T is fundamentally C^2 , γ_z is continuously differentiable on O , whose complement is countable. Furthermore,

$$\Delta \gamma_z(t) = \gamma_z(t) - \lim_{k \rightarrow \infty} \gamma_z(t - 1/k) = -(1-\beta) \Delta \lambda_n(\tilde{z} + te^n) \quad \forall t \in \mathbb{R}.$$

By Lemma 13, it follows that

$$\frac{\partial \hat{u}}{\partial \varepsilon}(z, \mu, \varepsilon) = \int_{\mathbb{R}} M \left(\left| \frac{t - \mu}{\varepsilon} \right| \right) \gamma'_z(t) dt + \sum_{t \in O^c} M \left(\left| \frac{t - \mu}{\varepsilon} \right| \right) \Delta \gamma_z(t) \quad \forall (z, \mu, \varepsilon) \in \mathcal{Z} \times \mathbb{R} \times (0, \infty).$$

Next, observe that

$$\begin{aligned}\mathbb{E}\Upsilon(\mu, \varepsilon) &= \int_{\mathbb{R}} \dots \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \Upsilon(x_1, \dots, x_{n-1}, x_n) \varphi_{\mu, \varepsilon}(x_n) dx_n \right) \tilde{\varphi}(x_1, \dots, x_{n-1}) dx_1 \dots dx_{n-1} \\ &= \int_{\mathbb{R}^{n-1}} \hat{u}(z, \mu, \varepsilon) \tilde{\varphi}(z) dz \quad \forall (\mu, \varepsilon) \in \mathbb{R} \times (0, \infty).\end{aligned}$$

By Lebesgue's Dominated Convergence Theorem and since $\mathcal{Z}^c$ has measure 0 and $\left| \frac{\hat{u}(z, \mu, \varepsilon + \delta) - \hat{u}(z, \mu, \varepsilon)}{\delta} \right| \leq 1$ for all $z \in \mathbb{R}^{n-1}$, for all $\delta \in (-\varepsilon, \infty) \setminus \{0\}$, and for all $\varepsilon \in (0, \infty)$, we can conclude that for each $(\mu, \varepsilon) \in \mathbb{R} \times (0, \infty)$

$$\frac{\partial \mathbb{E}\Upsilon}{\partial \varepsilon}(\mu, \varepsilon) = \int_{\mathbb{R}^{n-1}} \frac{\partial \hat{u}}{\partial \varepsilon}(z, \mu, \varepsilon) \tilde{\varphi}(z) dz.$$

By evenness and the unit-variance assumption, $\int_{\mathbb{R}} M(|(t - \mu)/\varepsilon|) dt = \varepsilon$. Since $\gamma'_z \geq -\beta/(1 - \beta)$ almost everywhere, $\Delta\gamma_z \geq 0$, and $|\partial_\varepsilon \hat{u}(z, \mu, \varepsilon)| \leq 1$, the slice identity implies

$$\int_{\mathbb{R}} M(|(t - \mu)/\varepsilon|) |\gamma'_z(t)| dt + \sum_{t \in \mathbb{R}} M(|(t - \mu)/\varepsilon|) \Delta\gamma_z(t) \leq 1 + \frac{2\beta}{1 - \beta} \varepsilon.$$

Thus both terms are separately integrable against $\tilde{\varphi}$. We set these terms equal to zero on the null set $\mathcal{Z}^c$. Consequently,

$$\begin{aligned}\frac{\partial \mathbb{E}\Upsilon}{\partial \varepsilon}(\mu, \varepsilon) &= \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} M\left(\left|\frac{t - \mu}{\varepsilon}\right|\right) \gamma'_z(t) dt \right) \tilde{\varphi}(z) dz + \int_{\mathbb{R}^{n-1}} \left(\sum_{t \in \mathbb{R}} M\left(\left|\frac{t - \mu}{\varepsilon}\right|\right) \Delta\gamma_z(t) \right) \tilde{\varphi}(z) dz.\end{aligned}$$

Substituting $\gamma'_z(t) = -(1 - \beta) \frac{\partial \lambda_n}{\partial x_n}(\tilde{z} + te^n)$ for almost every t , and $\Delta\gamma_z(t) = -(1 - \beta) \Delta\lambda_n(\tilde{z} + te^n)$ for every t , yields the formula in the statement. That $\Delta\lambda_n(x) \leq 0$ for every $x \in \mathbb{R}^n$ follows by Proposition 7. $\square$

Proof of Theorem 3. Fix $i \in [n]$ and $s \in \mathbb{R}_{++}^n$. After relabeling coordinates so that sector i is the last coordinate, apply Proposition 11 with

$$\varphi = \varphi_i, \quad \mu = \mu_i, \quad \varepsilon = s_i, \quad \tilde{\varphi} = \tilde{\varphi}_{-i, s}.$$

The integrability assumption on $\tilde{\varphi}$ holds because each $X_j = \mu_j + s_j \nu_j$ has a finite first absolute moment. Indeed,

$$\mathbb{E}_s \|X_{-i}\|_\infty \leq \sum_{j \neq i} (|\mu_j| + s_j \mathbb{E} |\nu_j|) < \infty.$$

Observe also that

$$M\left(\left|\frac{x_i - \mu_i}{s_i}\right|\right) = \int_{\left|\frac{x_i - \mu_i}{s_i}\right|}^{\infty} r \varphi_i(r) dr = m_i(x_i).$$

Finally, the sign follows from $\Delta\lambda_i(x) \leq 0$. $\square$

E Relaxing Cobb-Douglas in Labor

Suppose that the equilibrium log-price equation for each sector $i \in [n]$ can be written as

$$p_i = -\ln(Z_i) + T_i(p_0, p) \quad (162)$$

where $p_0 \in \mathbb{R}$ is the log-price of the primary input and $T_i : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$. This generalizes Equation 14 by allowing T_i to also depend on the price of labor. Next, suppose that there exist $\bar{\beta} \in (0, 1)$ and a normalized, non-decreasing, and translation invariant $S : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ such that

$$T_i(p_0, p) = (1 - \bar{\beta}) p_0 + \bar{\beta} S_i(p_0, p) \quad (163)$$

for all $(p_0, p) \in \mathbb{R}^{n+1}$ and $i \in [n]$. Assume also that S is a valid log-cost operator.²² The previous fixed-point equation then becomes

$$p_i = (1 - \bar{\beta})(x_i + p_0) + \bar{\beta} S_i(p_0, p) \quad \forall i \in [n] \quad (164)$$

where, as in the main text, $x_i = \frac{-\ln(Z_i)}{(1-\bar{\beta})}$ for all $i \in [n]$. Let $\Pi(p_0, x) \in \mathbb{R}^n$ denote the unique solution of the previous fixed-point equation indexed to $(p_0, x) \in \mathbb{R}^{n+1}$. This is well-defined because $y \mapsto (1 - \bar{\beta})(x + p_0 e) + \bar{\beta} S(p_0, y)$ is a contraction, hence admits a unique fixed point. In our main analysis, we always focus on $\Pi(0, x)$ since it is always possible to normalize $p_0 = 0$.

Consider the cloned economy where, for each original sector $i \in [n]$, we generate two sectors given by the original one i and a new cloned sector $i + n$. With this, the new cloned economy has $2n$ sectors. Define $\tilde{S} : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ as

$$\tilde{S}_j(\tilde{p}) = \begin{cases} S_j(\tilde{p}_{j+n}, \tilde{p}_1, \dots, \tilde{p}_n) & \text{if } j \in [n], \\ \tilde{p}_j & \text{if } j \in \{n+1, \dots, 2n\}. \end{cases} \quad (165)$$

Intuitively, to fit this more general dependence on labor in the baseline framework of the paper, for every sector i , we are adding an extra sector $i + n$ that transforms the household's labor supply into labor services sold to sector i (this is a fairly standard approach, see *e.g.*, La'O and Tahbaz-Salehi, 2022). It is immediate to see that $\tilde{S}$ is normalized, monotone, and translation invariant. Assume that $\tilde{S}$ represents the log-cost operator of our cloned economy.

The equilibrium price equation in the cloned economy parametrized to the log-cost of labor $p_0 \in \mathbb{R}$ and the cloned log-productivity $\tilde{x} \in \mathbb{R}^{2n}$ is

$$\tilde{p} = (1 - \bar{\beta})(\tilde{x} + p_0 e) + \bar{\beta} \tilde{S}(\tilde{p}). \quad (166)$$

Call $\tilde{\Pi}(p_0, \tilde{x}) \in \mathbb{R}^{2n}$ the unique solution of the previous fixed-point equation.

Lemma 14. *Define*

$$\tilde{x}_j = \begin{cases} x_j & \text{if } j \in [n], \\ 0 & \text{if } j \in \{n+1, \dots, 2n\}. \end{cases} \quad (167)$$

²²This is tantamount to assuming that the production function of sector i can be written as $\tilde{G}_i(Q_0, Q) = \max_{\ell \in [0, Q_0]} ((Q_0 - \ell)/(1 - \bar{\beta}))^{1-\bar{\beta}} (\hat{G}_i(\ell, Q)/\bar{\beta})^{\bar{\beta}}$, where $\hat{G}_i : \mathbb{R}_+^{n+1} \rightarrow \mathbb{R}$ is positive homogeneous, non-decreasing, continuous, and with $\max_{\{Q_0, Q: Q_0 + \sum_{i=1}^n Q_i = 1\}} \hat{G}_i(Q_0, Q) = 1$.

We have

$$\tilde{\Pi}_j(0, \tilde{x}) = \begin{cases} \Pi_j(0, x) & \text{if } j \in [n], \\ 0 & \text{if } j \in \{n+1, \dots, 2n\}. \end{cases} \quad (168)$$

Proof. That $\tilde{\Pi}_j(0, \tilde{x}) = 0$ for all $j \in \{n+1, \dots, 2n\}$ follows from

$$\tilde{\Pi}_j(0, \tilde{x}) = (1 - \bar{\beta})(\tilde{x}_j + 0) + \bar{\beta}\tilde{S}_j(\tilde{\Pi}(0, \tilde{x})) = \bar{\beta}\tilde{\Pi}_j(0, \tilde{x}) \quad \forall j \in \{n+1, \dots, 2n\}. \quad (169)$$

With this, we have that for all $j \in [n]$

$$\tilde{\Pi}_j(0, \tilde{x}) = (1 - \bar{\beta})(\tilde{x}_j + 0) + \bar{\beta}\tilde{S}_j(\tilde{\Pi}(0, \tilde{x})) = (1 - \bar{\beta})x_j + \bar{\beta}S_j(0, \tilde{\Pi}(0, \tilde{x})_1, \dots, \tilde{\Pi}(0, \tilde{x})_n). \quad (170)$$

Since this coincides with the system of equations in (164) that defines $\Pi_j(0, x)$ and it admits a unique solution, we obtain $\tilde{\Pi}_j(0, \tilde{x}) = \Pi_j(0, x)$ for all $j \in [n]$. $\square$

Crucially, the economy described by the log-cost operator of Equation (165) satisfies our Assumption 1, and therefore, all our results can be applied to it verbatim. Given the relation established between the prices of the cloned and original economy, these results are easily transferred to the latter, too.

F Producer Theory

We define by $\mathcal{G}$ the collection of all production functions $G : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ that are nondegenerate,²³ monotone, upper semicontinuous, and positive homogeneous.²⁴ We denote by $\mathcal{G}_{\text{nor}}$ the collection of all production functions $G : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ that are monotone, upper semicontinuous, positive homogeneous, and such that $\max_{w \in \Delta} G(w) = 1$. Clearly, we have that $\mathcal{G}_{\text{nor}} \subseteq \mathcal{G}$. We define by $\mathcal{G}_{\text{con}}$ (resp. $\mathcal{G}_{\text{qcon}}$) the subset of functions in $\mathcal{G}$ that are concave (resp. quasiconcave). We set $\mathcal{G}_{\text{nor-con}} = \mathcal{G}_{\text{nor}} \cap \mathcal{G}_{\text{con}}$. Given $G \in \mathcal{G}$, we define the expenditure function of G at 1, denoted $K_G : \mathbb{R}_{++}^n \rightarrow \mathbb{R}_{++}$, by

$$K_G(P) = \min_{Q \in \mathbb{R}_+^n : G(Q) \geq 1} \sum_{j=1}^n Q_j P_j \quad \forall P \in \mathbb{R}_{++}^n.$$

We will show that indeed K_G is well defined (cf. Lemma 15). We will also show that K_G is monotone, continuous, concave, and positive homogeneous. We denote the class of functions $K : \mathbb{R}_{++}^n \rightarrow \mathbb{R}_{++}$ with these four properties by $\mathcal{K}$. We also denote by $\mathcal{K}_{\text{nor}}$ the subset of functions of $\mathcal{K}$ that are such that $K(e) = 1$. A goal of this appendix is to show that $G \mapsto K_G$ is a bijection from $\mathcal{G}_{\text{nor-con}}$ onto $\mathcal{K}_{\text{nor}}$. Finally, given a function $K \in \mathcal{K}$, we define $\kappa : \mathbb{R}^n \rightarrow \mathbb{R}$ by $\kappa(p) = \ln(K(\exp(p)))$ for all $p \in \mathbb{R}^n$ where $\exp(p)$ is the vector whose j -th component is e^{p_j} for all $j \in [n]$. We conclude the paper by characterizing convexity and concavity of κ in terms of G . Throughout the note, we assume that $0 \cdot \pm\infty = 0$, $0/0 = 0$, $\infty^0 = 1$, $0^0 = 1$, $\ln \infty = \infty$, and $\ln 0 = -\infty$. Given a nonempty set X and a function $f : X \rightarrow \mathbb{R}$ and $t \in \mathbb{R}$, we denote by

²³That is, $G(e) > 0$.

²⁴That is, $G(\lambda Q) = \lambda G(Q)$ for all $\lambda > 0$ and for all $Q \in \mathbb{R}_+^n$.

$\{f \geq t\} = \{x \in X : f(x) \geq t\}$ the upper contour set of f at t . Similarly, we define the lower contour set $\{f \leq t\}$.

Remark 7. In what follows, whenever $G \in \mathcal{G}$, it will automatically be granted that G is continuous by the classical result of [Moore \(1999\)](#). $\triangle$

F.1 Expenditure functions

We move to study the expenditure functions induced by our class of production functions. Given $G \in \mathcal{G}$, recall that $K_G : \mathbb{R}_{++}^n \rightarrow \mathbb{R}_{++}$ is defined by

$$K_G(P) = \min_{Q \in \mathbb{R}_+^n : G(Q) \geq 1} \sum_{j=1}^n Q_j P_j \quad \forall P \in \mathbb{R}_{++}^n.$$

We also define $U_G : \mathbb{R}_+ \times \mathbb{R}_+^n \rightarrow [0, \infty]$ and $U_{G,\text{int}} : \mathbb{R}_{++} \times \mathbb{R}_+^n \rightarrow [0, \infty]$ by

$$U_G(t, P) = \sup \{G(Q) : Q \in \mathbb{R}_+^n \text{ and } \langle Q, P \rangle \leq t\} \quad \forall (t, P) \in \mathbb{R}_+ \times \mathbb{R}_+^n$$

and

$$U_{G,\text{int}}(t, P) = \sup \{G(Q) : Q \in \mathbb{R}_{++}^n \text{ and } \langle Q, P \rangle \leq t\} \quad \forall (t, P) \in \mathbb{R}_{++} \times \mathbb{R}_+^n.$$

We next show that expenditure functions inherit most of the functional properties defining the functions in $\mathcal{G}$. In particular, the next lemma shows that if $G \in \mathcal{G}$ (resp. $\mathcal{G}_{\text{nor}}$), then $K_G \in \mathcal{K}$ (resp. $\mathcal{K}_{\text{nor}}$).

Lemma 15. *If $G \in \mathcal{G}$ (resp. $G \in \mathcal{G}_{\text{nor}}$), then K_G is well defined and belongs to $\mathcal{K}$ (resp. $\mathcal{K}_{\text{nor}}$).*

Proof. Since G is continuous and positive homogeneous, we have that $G(0) = 0$. Fix $P \in \mathbb{R}_{++}^n$. Since G is nondegenerate and positive homogeneous, denote by $\hat{Q}$ an element in $\mathbb{R}_+^n$ such that $G(\hat{Q}) = 1$. Since G is monotone, continuous, positive homogeneous, and such that $G(0) = 0$, we have that

$$\begin{aligned} \inf_{Q \in \mathbb{R}_+^n : G(Q) \geq 1} \sum_{j=1}^n Q_j P_j &= \inf_{Q \in \mathbb{R}_+^n : G(Q) = 1} \sum_{j=1}^n Q_j P_j = \inf_{Q \in \mathbb{R}_+^n : G(Q) = 1 \text{ and } \sum_{j=1}^n Q_j P_j \leq \sum_{j=1}^n \hat{Q}_j P_j} \sum_{j=1}^n Q_j P_j \\ &= \min_{Q \in \mathbb{R}_+^n : G(Q) = 1 \text{ and } \sum_{j=1}^n Q_j P_j \leq \sum_{j=1}^n \hat{Q}_j P_j} \sum_{j=1}^n Q_j P_j > 0, \end{aligned}$$

proving that the inf is attained and K_G is well defined. Next, for each $Q \in \{G \geq 1\}$ define $\ell_Q : \mathbb{R}_{++}^n \rightarrow \mathbb{R}_{++}$ by $\ell_Q(P) = \langle Q, P \rangle$ for all $P \in \mathbb{R}_{++}^n$. Since $K_G = \inf_{Q \in \{G \geq 1\}} \ell_Q$ and each ℓ_Q is monotone, continuous, affine, and positive homogeneous, we have that K_G is monotone, concave, and positive homogeneous. By ([Borwein and Lewis, 2006](#), Theorem 4.1.3) and since K_G is concave, it follows that K_G is continuous, proving the main statement. Finally, if $\max_{w \in \Delta} G(w) = 1$, then $\hat{Q} = \ddot{w}$ can be chosen to be in Δ . This implies that $1 = \sum_{j=1}^n \ddot{w}_j = \sum_{j=1}^n \ddot{w}_j e_j \geq K_G(e)$ where $\ddot{w} \in \Delta$ is such that $G(\ddot{w}) = 1$. Next, consider $Q \in \mathbb{R}_+^n$ such that $G(Q) \geq 1$ and $\sum_{j=1}^n Q_j = K_G(e) \leq 1$. Since $G(Q) \geq 1$, it follows that $Q \neq 0$ and $\sum_{j=1}^n Q_j = \|Q\|_1 \in (0, 1]$. Since $\max_{w \in \Delta} G(w) = 1$ and $G(Q) \geq 1$, this implies that $\hat{w} = Q/\|Q\|_1 \in \Delta$ is such that

$1 \geq G(\hat{w}) = G(Q) / \|Q\|_1 \geq 1$, proving that $G(Q) / \|Q\|_1 = 1$, that is, $1 \leq G(Q) = \|Q\|_1 \leq 1$. It follows that $K_G(e) = \sum_{j=1}^n Q_j = \|Q\|_1 = 1$, proving the statement. $\square$

We next study some of the properties of the indirect utility U_G and relate it to K_G . Those are classic results. We begin with an ancillary fact.

Lemma 16. *If $G \in \mathcal{G}$, then $U_G(t, P) \in (0, \infty)$ for all $(t, P) \in \mathbb{R}_{++} \times \mathbb{R}_{++}^n$ and $U_G(t, P) = U_{G,\text{int}}(t, P)$ for all $(t, P) \in \mathbb{R}_{++} \times \mathbb{R}_+^n \setminus \{0\}$.*

Proof. Consider $(t, P) \in \mathbb{R}_{++} \times \mathbb{R}_{++}^n$ and set $\lambda = \left(\sum_{j=1}^n P_j\right)^{-1} t \in (0, \infty)$ and $Q = \lambda e \in \mathbb{R}_{++}^n$. Since G is nondegenerate and positive homogeneous, we have that $\langle Q, P \rangle = t$ and $0 < \lambda G(e) = G(\lambda e) = G(Q) \leq U_G(t, P)$. At the same time, since $(t, P) \in \mathbb{R}_{++} \times \mathbb{R}_{++}^n$, we have that $\{Q \in \mathbb{R}_+^n : \langle Q, P \rangle \leq t\}$ is closed and bounded and, in particular, compact. Since G is upper semicontinuous and real-valued, it follows that the sup defining $U_G(t, P)$ is attained and real-valued too. Since (t, P) was arbitrarily chosen, the first part of the statement follows. Consider $(t, P) \in \mathbb{R}_{++} \times \mathbb{R}_+^n \setminus \{0\}$. Since $P \in \mathbb{R}_+^n \setminus \{0\}$, define $\sigma = \sum_{j=1}^n P_j > 0$. Clearly, we have that $\{Q \in \mathbb{R}_+^n : \langle Q, P \rangle \leq t\}$ is a closed set and, in particular, $\text{cl}\{Q \in \mathbb{R}_{++}^n : \langle Q, P \rangle \leq t\} \subseteq \{Q \in \mathbb{R}_+^n : \langle Q, P \rangle \leq t\}$. Consider $Q \in \mathbb{R}_+^n$ such that $\langle Q, P \rangle \leq t$. We have two cases: either $\langle Q, P \rangle < t$ or $\langle Q, P \rangle = t$. In the first case, define $s = t - \langle Q, P \rangle > 0$ and for each $k \in \mathbb{N}$ define $Q^k = Q + \frac{s}{\sigma k} e \in \mathbb{R}_{++}^n$. Note that $\langle Q^k, P \rangle = \langle Q, P \rangle + \frac{s}{k} \leq \langle Q, P \rangle + s = t$ for all $k \in \mathbb{N}$ and $\lim_k Q^k = Q$, proving that $Q \in \text{cl}\{Q \in \mathbb{R}_{++}^n : \langle Q, P \rangle \leq t\}$. In the second case, since $t > 0$ and $P \in \mathbb{R}_+^n \setminus \{0\}$, the condition $\langle Q, P \rangle = t$ implies that $Q_{\bar{j}} P_{\bar{j}} > 0$ for some $\bar{j} \in [n]$ and, in particular, $Q_{\bar{j}} > 0$ and $P_{\bar{j}} > 0$ for some $\bar{j} \in [n]$. For each $k \in \mathbb{N}$ define $s_k = \frac{1}{k+1} Q_{\bar{j}} P_{\bar{j}} > 0$ and define $Q^k = Q - \frac{1}{k+1} Q_{\bar{j}} e^{\bar{j}} + \frac{s_k}{\sigma} e \in \mathbb{R}_{++}^n$. Note that $\langle Q^k, P \rangle = \langle Q, P \rangle - s_k + s_k = \langle Q, P \rangle = t$ for all $k \in \mathbb{N}$ and $\lim_k Q^k = Q$, proving that $Q \in \text{cl}\{Q \in \mathbb{R}_{++}^n : \langle Q, P \rangle \leq t\}$. Since Q was arbitrarily chosen, we have that $\{Q \in \mathbb{R}_+^n : \langle Q, P \rangle \leq t\} \subseteq \text{cl}\{Q \in \mathbb{R}_{++}^n : \langle Q, P \rangle \leq t\}$ and, in particular, $\{Q \in \mathbb{R}_+^n : \langle Q, P \rangle \leq t\} = \text{cl}\{Q \in \mathbb{R}_{++}^n : \langle Q, P \rangle \leq t\}$. Since G is continuous, we have that

$$U_G(t, P) = \sup_{Q \in \mathbb{R}_+^n : \langle Q, P \rangle \leq t} G(Q) = \sup_{Q \in \mathbb{R}_{++}^n : \langle Q, P \rangle \leq t} G(Q) = U_{G,\text{int}}(t, P).$$

Since (t, P) was arbitrarily chosen, the second part of the statement follows. $\square$

Lemma 17. *If $G \in \mathcal{G}$, then $U_G(K_G(P), P) = 1$ for all $P \in \mathbb{R}_{++}^n$.*

Proof. Fix $P \in \mathbb{R}_{++}^n$. Consider $\bar{Q} \in \mathbb{R}_+^n$ such that $G(\bar{Q}) \geq 1$ and $K_G(P) = \sum_{j=1}^n \bar{Q}_j P_j$. Since $\sum_{j=1}^n \bar{Q}_j P_j \leq K_G(P)$, we have that $U_G(K_G(P), P) \geq G(\bar{Q}) \geq 1$. By contradiction, assume that $U_G(K_G(P), P) > 1$. Since G is upper semicontinuous and $P \in \mathbb{R}_{++}^n$, consider $\hat{Q} \in \mathbb{R}_+^n$ such that $\sum_{j=1}^n \hat{Q}_j P_j \leq K_G(P)$ and $G(\hat{Q}) = U_G(K_G(P), P) > 1$. By definition of K_G and since $G(\hat{Q}) > 1$, this would imply that $\sum_{j=1}^n \hat{Q}_j P_j \geq K_G(P)$, yielding that $K_G(P) = \sum_{j=1}^n \hat{Q}_j P_j$. Define $\gamma = 1/G(\hat{Q}) \in (0, 1)$ and $\tilde{Q} = \gamma \hat{Q} \in \mathbb{R}_+^n$. Since G is positive homogeneous and $\sum_{j=1}^n \hat{Q}_j P_j = K_G(P) > 0$, we would have that $G(\tilde{Q}) = 1$ and thus a contradiction:

$$\sum_{j=1}^n \tilde{Q}_j P_j = \gamma \sum_{j=1}^n \hat{Q}_j P_j < \sum_{j=1}^n \hat{Q}_j P_j = K_G(P) \leq \sum_{j=1}^n \tilde{Q}_j P_j.$$

$\square$

F.2 Expenditure functions and lattice operations

We next explore the question of whether lattice operations on production functions translate into lattice operations over expenditure functions. This will play a role in characterizing when a (log-expenditure) function κ is either convex or concave. We first show that the expenditure function of a production function which is the infimum of production functions is the maximum of the corresponding expenditure functions, provided some extra conditions are met. To this end, given a function from a nonempty set C of a normed vector space to the real line, we say that it is lower (resp. upper) coercive if and only if its lower (resp. upper) contour sets are compact.

Lemma 18. *Let $\tilde{C}$ be a nonempty precompact convex subset of a normed vector space and $\mathfrak{G} : \tilde{C} \times \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ be such that $w \mapsto \mathfrak{G}(w, Q)$ is lower coercive and quasiconvex for all $Q \in \mathbb{R}_{++}^n$ and $\mathfrak{G}(w, \cdot) \in \mathcal{G}_{\text{con}}$ for all $w \in \tilde{C}$. If $G : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ is such that*

$$G(Q) = \inf_{w \in \tilde{C}} \mathfrak{G}(w, Q) \quad \forall Q \in \mathbb{R}_+^n,$$

then the inf is attained on $\mathbb{R}_{++}^n$ and G is well defined and belongs to $\mathcal{G}_{\text{con}}$. Moreover, we have that $K_G = \max_{w \in \tilde{C}} K_w$ where $K_w = K_{\mathfrak{G}(w, \cdot)}$.

Proof. Since $Q \mapsto \mathfrak{G}(w, Q)$ is nonnegative, monotone, upper semicontinuous, concave, and positive homogeneous for all $w \in \tilde{C}$, it is plain that G is well defined, monotone, upper semicontinuous, concave, and positive homogeneous. By Remark 7, G is continuous. Since $w \mapsto \mathfrak{G}(w, Q)$ is lower coercive for all $Q \in \mathbb{R}_{++}^n$, we have that $G(Q) = \min_{w \in \tilde{C}} \mathfrak{G}(w, Q)$ for all $Q \in \mathbb{R}_{++}^n$. In particular, this implies that G is nondegenerate and, in particular, $G \in \mathcal{G}$. Define $\bar{\mathfrak{G}} : \text{cl } \tilde{C} \times \mathbb{R}_+^n \rightarrow [0, \infty]$ by

$$\bar{\mathfrak{G}}(w, Q) = \begin{cases} \mathfrak{G}(w, Q) & \text{if } w \in \tilde{C} \\ \infty & \text{if } w \notin \tilde{C} \end{cases} \quad \forall (w, Q) \in \text{cl } \tilde{C} \times \mathbb{R}_+^n.$$

By definition of $\bar{\mathfrak{G}}$ and since $w \mapsto \mathfrak{G}(w, Q)$ is lower coercive and quasiconvex for all $Q \in \mathbb{R}_{++}^n$ on $\tilde{C}$, we have that $w \mapsto \bar{\mathfrak{G}}(w, Q)$ is lower semicontinuous and quasiconvex on $\text{cl } \tilde{C}$ for all $Q \in \mathbb{R}_{++}^n$. Similarly, $Q \mapsto \bar{\mathfrak{G}}(w, Q)$ is monotone, continuous, concave, and positive homogeneous on $\mathbb{R}_+^n$ for all $w \in \text{cl } \tilde{C}$. Fix $P \in \mathbb{R}_{++}^n$. Note that if $Q \in \mathbb{R}_+^n$ and $G(Q) \geq 1$, then $\mathfrak{G}(w, Q) \geq 1$ for all $w \in \tilde{C}$, proving that

$$\sum_{j=1}^n Q_j P_j \geq K_w(P) \quad \forall w \in \tilde{C} \implies \sum_{j=1}^n Q_j P_j \geq \sup_{w \in \tilde{C}} K_w(P) \implies K_G(P) \geq \sup_{w \in \tilde{C}} K_w(P).$$

Define $U_w = U_{\mathfrak{G}(w, \cdot)}$ and $U_{w, \text{int}} = U_{\bar{\mathfrak{G}}(w, \cdot), \text{int}}$ for all $w \in \tilde{C}$. Observe that $G(Q) = \min_{w \in \tilde{C}} \mathfrak{G}(w, Q) = \min_{w \in \text{cl } \tilde{C}} \bar{\mathfrak{G}}(w, Q) = \min_{w \in \text{cl } \tilde{C}} \bar{\mathfrak{G}}(w, Q)$ for all $Q \in \mathbb{R}_{++}^n$. By Sion's Maxmin Theorem and Lemma 16 and since $w \mapsto \bar{\mathfrak{G}}(w, Q)$ is lower semicontinuous and quasiconvex for all $Q \in \mathbb{R}_{++}^n$

and $Q \mapsto \bar{\mathfrak{G}}(w, Q)$ is upper semicontinuous and quasiconcave for all $w \in \text{cl } \tilde{C}$, note that

$$\begin{aligned} U_G(t, P) &= U_{G, \text{int}}(t, P) = \sup_{Q \in \mathbb{R}_{++}^n : \sum_{j=1}^n Q_j P_j \leq t} \inf_{w \in \text{cl } \tilde{C}} \bar{\mathfrak{G}}(w, Q) = \inf_{w \in \text{cl } \tilde{C}} \sup_{Q \in \mathbb{R}_{++}^n : \sum_{j=1}^n Q_j P_j \leq t} \bar{\mathfrak{G}}(w, Q) \\ &= \min_{w \in \text{cl } \tilde{C}} \sup_{Q \in \mathbb{R}_{++}^n : \sum_{j=1}^n Q_j P_j \leq t} \bar{\mathfrak{G}}(w, Q) = \min_{w \in \tilde{C}} \sup_{Q \in \mathbb{R}_{++}^n : \sum_{j=1}^n Q_j P_j \leq t} \bar{\mathfrak{G}}(w, Q) \\ &= \min_{w \in \tilde{C}} U_{w, \text{int}}(t, P) = \min_{w \in \tilde{C}} U_w(t, P) \quad \forall t > 0, \end{aligned}$$

proving that $U_G(t, P) = \min_{w \in \tilde{C}} U_w(t, P)$ for all $t > 0$. By Lemma 17, we have that $1 = U_G(K_G(P), P) = \min_{w \in \tilde{C}} U_w(K_G(P), P)$ and $U_w(K_w(P), P) = 1$ for all $w \in \tilde{C}$ and, in particular, there exists $\tilde{w} \in \tilde{C}$ such that $1 = U_G(K_G(P), P) = U_{\tilde{w}}(K_G(P), P)$. Since $P \in \mathbb{R}_{++}^n$ and $\mathfrak{G}(\tilde{w}, \cdot)$ is upper semicontinuous, this implies that $U_{\tilde{w}}(1, P) \in (0, \infty)$. Since $U_{\tilde{w}}(K_{\tilde{w}}(P), P) = 1$ and $t \mapsto U_{\tilde{w}}(t, P)$ is real-valued and positive homogeneous, it follows that

$$K_{\tilde{w}}(P) U_{\tilde{w}}(1, P) = U_{\tilde{w}}(K_{\tilde{w}}(P), P) = 1 = U_{\tilde{w}}(K_G(P), P) = K_G(P) U_{\tilde{w}}(1, P),$$

proving that $K_G(P) = K_{\tilde{w}}(P) \leq \sup_{w \in \tilde{C}} K_w(P)$. Since $K_G(P) \geq \sup_{w \in \tilde{C}} K_w(P)$ and P was arbitrarily chosen, we can conclude that

$$K_G(P) = \max_{w \in \tilde{C}} K_w(P) \quad \forall P \in \mathbb{R}_{++}^n,$$

proving the statement. $\square$

We next show that the expenditure function of a production function which is the maximum of production functions is the minimum of the corresponding expenditure functions.

Lemma 19. *Let $\tilde{C}$ be a nonempty precompact subset of a metric space and $\mathfrak{G} : \text{cl } \tilde{C} \times \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ be upper semicontinuous and such that $\mathfrak{G}(w, \cdot) \in \mathcal{G}$ for all $w \in \tilde{C}$ and $\mathfrak{G}(w, \cdot) = 0$ for all $w \in \text{cl } \tilde{C} \setminus \tilde{C}$. If $G : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ is defined by*

$$G(Q) = \sup_{w \in \tilde{C}} \mathfrak{G}(w, Q) \quad \forall Q \in \mathbb{R}_+^n, \quad (171)$$

then the sup is attained on $\mathbb{R}_+^n$ and G is well defined and belongs to $\mathcal{G}$. Moreover, we have that:

1. $K_G = \min_{w \in \tilde{C}} K_w$ where $K_w = K_{\mathfrak{G}(w, \cdot)}$ for all $w \in \tilde{C}$.
2. If $\tilde{C}$ is a convex subset of a normed vector space and $\tilde{w} \mapsto U_{\mathfrak{G}(\tilde{w}, \cdot)}(t, w)$ is upper semicontinuous and quasiconcave on $\tilde{C}$ for all $(t, w) \in \mathbb{R}_{++} \times \Delta$ and $\mathfrak{G}(\tilde{w}, \cdot) \in \mathcal{G}_{\text{con}}$ for all $\tilde{w} \in \tilde{C}$, then $G \in \mathcal{G}_{\text{con}}$.

Proof. Since $Q \mapsto \mathfrak{G}(w, Q)$ is nonnegative, monotone, continuous, and positive homogeneous for all $w \in \tilde{C}$, it is plain that G is nonnegative, monotone, lower semicontinuous, and positive homogeneous. Since $\mathfrak{G}(w, e) > 0$ for all $w \in \tilde{C}$, it follows that $G(e) > 0$, that is, G is nondegenerate. Since $\mathfrak{G} \geq 0$ and $\mathfrak{G}(w, \cdot) = 0$ for all $w \in \text{cl } \tilde{C} \setminus \tilde{C}$, the sup over $\tilde{C}$ in (171) can be replaced with the sup over $\text{cl } \tilde{C}$. By (Aliprantis and Border, 2006, Lemma 17.30) and since $\mathfrak{G} : \text{cl } \tilde{C} \times \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ is upper semicontinuous and $\text{cl } \tilde{C}$ is compact, this implies that G is real-valued and upper semicontinuous and the sup is attained on $\mathbb{R}_+^n$ and, in particular, G is well defined and belongs to $\mathcal{G}$.

1. Fix $P \in \mathbb{R}_{++}^n$ and $w \in \tilde{C}$. Since $\mathfrak{G}(w, Q) \geq 1$ implies that $G(Q) \geq 1$, it follows that

$$K_w(P) = \min_{Q \in \mathbb{R}_+^n: \mathfrak{G}(w, Q) \geq 1} \sum_{j=1}^n Q_j P_j \geq \min_{Q \in \mathbb{R}_+^n: G(Q) \geq 1} \sum_{j=1}^n Q_j P_j = K_G(P).$$

Since P and w were arbitrarily chosen, we can conclude that $K_w \geq K_G$ for all $w \in \tilde{C}$ and $\inf_{w \in \tilde{C}} K_w \geq K_G$. Fix $P \in \mathbb{R}_{++}^n$. By Lemma 15, consider $Q \in \mathbb{R}_+^n$ such that $G(Q) \geq 1$ and $K_G(P) = \sum_{j=1}^n Q_j P_j$. By definition of G , there exists $\bar{w} \in \tilde{C}$ such that $\mathfrak{G}(\bar{w}, Q) \geq 1$ and, in particular,

$$\inf_{w \in \tilde{C}} K_w(P) \geq K_G(P) = \sum_{j=1}^n Q_j P_j \geq K_{\bar{w}}(P) \geq \inf_{w \in \tilde{C}} K_w(P),$$

proving that $K_G(P) = \min_{w \in \tilde{C}} K_w(P)$. Since P was arbitrarily chosen, the statement follows.

2. Given G in (171), define $\bar{G} : \mathbb{R}^n \rightarrow [-\infty, \infty]$ by $\bar{G}(Q) = -\infty$ for all $Q \notin \mathbb{R}_{++}^n$ and $\bar{G}(Q) = G(Q)$ for all $Q \in \mathbb{R}_{++}^n$. Since G is continuous, we have that $\bar{G}$ is lower semicontinuous. Define $H_{\bar{G}} : \mathbb{R} \times \Delta \rightarrow [-\infty, \infty]$ by

$$H_{\bar{G}}(t, w) = \sup_{Q \in \mathbb{R}^n: \langle w, Q \rangle \leq t} \bar{G}(Q) \quad \forall (t, w) \in \mathbb{R} \times \Delta.$$

Note that $H_{\bar{G}}(t, w) = U_{G, \text{int}}(t, w)$ for all $(t, w) \in \mathbb{R}_{++} \times \Delta$. Define $\tilde{G} : \mathbb{R}_{++}^n \rightarrow \mathbb{R}_+$ by

$$\tilde{G}(Q) = \inf_{w \in \Delta} H_{\bar{G}}(\langle w, Q \rangle, w) \quad \forall Q \in \mathbb{R}_{++}^n.$$

Since $t \mapsto H_{\bar{G}}(t, w)$ is monotone and $Q \mapsto \langle w, Q \rangle$ is affine for all $w \in \Delta$, it is immediate to see that $Q \mapsto H_{\bar{G}}(\langle w, Q \rangle, w)$ is quasiconcave on $\mathbb{R}_{++}^n$ for all $w \in \Delta$ and, in particular, so is $\tilde{G}$. Fix $\tilde{w} \in \tilde{C}$. Define $\tilde{\mathfrak{G}}(\tilde{w}, \cdot) : \mathbb{R}^n \rightarrow [-\infty, \infty]$ by $\tilde{\mathfrak{G}}(\tilde{w}, Q) = -\infty$ for all $Q \notin \mathbb{R}_{++}^n$ and $\tilde{\mathfrak{G}}(\tilde{w}, Q) = \tilde{G}(Q)$ for all $Q \in \mathbb{R}_{++}^n$. Since $\tilde{\mathfrak{G}}(\tilde{w}, \cdot)$ is monotone, continuous, and quasiconcave, we have that $\tilde{\mathfrak{G}}(\tilde{w}, \cdot)$ is monotone, lower semicontinuous, and quasiconcave. Define $H_{\tilde{w}} : \mathbb{R} \times \Delta \rightarrow [-\infty, \infty]$ by

$$H_{\tilde{w}}(t, w) = \sup_{Q \in \mathbb{R}^n: \langle w, Q \rangle \leq t} \tilde{\mathfrak{G}}(\tilde{w}, Q) \quad \forall (t, w) \in \mathbb{R} \times \Delta.$$

Define $U_{\tilde{w}, \text{int}} = U_{\tilde{\mathfrak{G}}(\tilde{w}, \cdot), \text{int}}$ on $\mathbb{R}_{++} \times \Delta$. By definition of $\tilde{\mathfrak{G}}(\tilde{w}, \cdot)$, we have that $H_{\tilde{w}}(t, w) = U_{\tilde{\mathfrak{G}}(\tilde{w}, \cdot), \text{int}}(t, w) = U_{\tilde{w}, \text{int}}(t, w)$ for all $(t, w) \in \mathbb{R}_{++} \times \Delta$. By Cerreia-Vioglio et al. (2011) and since $\tilde{w}$ was arbitrarily chosen,²⁵ we have that $H_{\tilde{w}}$ is lower semicontinuous and quasiconvex on $\mathbb{R} \times \Delta$ for all $\tilde{w} \in \tilde{C}$ and, in particular, so is $U_{\tilde{w}, \text{int}}$ on $\mathbb{R}_{++} \times \Delta$ for all $\tilde{w} \in \tilde{C}$ and

$$\mathfrak{G}(\tilde{w}, Q) = \tilde{\mathfrak{G}}(\tilde{w}, Q) = \min_{w \in \Delta} H_{\tilde{w}}(\langle w, Q \rangle, w) = \min_{w \in \Delta} U_{\tilde{w}, \text{int}}(\langle w, Q \rangle, w) \quad \forall Q \in \mathbb{R}_{++}^n, \forall \tilde{w} \in \tilde{C}.$$

By (171), we have that $U_{G, \text{int}} = \sup_{\tilde{w} \in \tilde{C}} U_{\tilde{w}, \text{int}}$. Fix $Q \in \mathbb{R}_{++}^n$. Define the function $F_Q : \Delta \times \tilde{C} \rightarrow [0, \infty]$ by $F_Q(w, \tilde{w}) = U_{\tilde{w}, \text{int}}(\langle w, Q \rangle, w)$ for all $(w, \tilde{w}) \in \Delta \times \tilde{C}$. Since $U_{\tilde{w}, \text{int}}$ is lower semicontinuous and quasiconvex on $\mathbb{R}_{++} \times \Delta$ for all $\tilde{w} \in \tilde{C}$, it follows that F_Q is lower semicontinuous and quasiconvex in the first argument. By assumption, F_Q is upper semicontinuous and quasicon-

²⁵See the working paper version (Cerreia-Vioglio et al., 2011, Lemmas 4 and 5, Theorem 1, and p.5).

cave in the second argument. By definition of F_Q and Sion's Maxmin Theorem (see Corollary 3.3 in [Sion, 1958](#)) and since F_Q is lower semicontinuous and quasiconvex in the first argument and upper semicontinuous and quasiconcave in the second argument and Δ is compact and convex and $\tilde{C}$ is convex, we have that

$$\begin{aligned}\tilde{G}(Q) &= \inf_{w \in \Delta} U_{G, \text{int}}(\langle w, Q \rangle, w) = \inf_{w \in \Delta} \sup_{\tilde{w} \in \tilde{C}} U_{\tilde{w}, \text{int}}(\langle w, Q \rangle, w) = \inf_{w \in \Delta} \sup_{\tilde{w} \in \tilde{C}} F_Q(w, \tilde{w}) \\ &= \sup_{\tilde{w} \in \tilde{C}} \inf_{w \in \Delta} F_Q(w, \tilde{w}) = \sup_{\tilde{w} \in \tilde{C}} \inf_{w \in \Delta} U_{\tilde{w}, \text{int}}(\langle w, Q \rangle, w) = \sup_{\tilde{w} \in \tilde{C}} \mathfrak{G}(\tilde{w}, Q) = G(Q).\end{aligned}$$

Since Q was arbitrarily chosen, we have that G coincides with $\tilde{G}$ on $\mathbb{R}_{++}^n$, proving that G is quasiconcave on $\mathbb{R}_{++}^n$. Since G is continuous, it follows that G is quasiconcave on $\mathbb{R}_+^n$. Since G is positive homogeneous, this implies that $G \in \mathcal{G}_{\text{con}}$, proving the statement. $\square$

F.3 Representation result

We can now move to our main representation result of this appendix. We provide a representation of the expenditure function in terms of its generating production function. The main representation result will follow and build on this and provide a canonical representation in terms of an alternative and *unique concave* production function.

Lemma 20. *If $G \in \mathcal{G}$, then*

$$K_G(P) = \min_{w \in \Delta} \frac{\langle w, P \rangle}{G(w)} \quad \forall P \in \mathbb{R}_{++}^n.$$

Moreover, $K_G(e) = 1$ if and only if $\max_{w \in \Delta} G(w) = 1$ (that is $G \in \mathcal{G}_{\text{nor}}$).

Proof. Fix $P \in \mathbb{R}_{++}^n$ and $w \in \Delta$ such that $G(w) \neq 0$. Define $Q = \frac{1}{G(w)}w \in \mathbb{R}_+^n$. Since G is positive homogeneous, we have that $G(Q) = 1$. By definition of Q and K_G , it follows that

$$\frac{\langle w, P \rangle}{G(w)} = \sum_{j=1}^n Q_j P_j \geq K_G(P).$$

Since w was arbitrarily chosen, it follows that $\min_{w \in \Delta} \frac{\langle w, P \rangle}{G(w)} \geq \min_{w \in \Delta: G(w) \neq 0} \frac{\langle w, P \rangle}{G(w)} \geq K_G(P)$. Next, fix $Q \in \mathbb{R}_+^n$ such that $G(Q) \geq 1$. Since G is nondegenerate and positive homogeneous, this implies that $\sum_{j=1}^n Q_j > 0$.²⁶ Define $w = \frac{1}{\sum_{j=1}^n Q_j} Q \in \Delta$. Since $G(Q) \geq 1$, it follows that

$$\sum_{j=1}^n Q_j P_j \geq \frac{\sum_{j=1}^n Q_j P_j}{G(Q)} = \frac{\sum_{j=1}^n w_j P_j}{G(w)}$$

where the equality follows from positive homogeneity of G . Because Q was arbitrarily chosen, it follows that $K_G(P) \geq \min_{w \in \Delta} \frac{\langle w, P \rangle}{G(w)}$, proving the main statement. Finally, observe that $K_G(e) = 1$ if and only if $\min_{w \in \Delta} \frac{1}{G(w)} = 1$ if and only if $\max_{w \in \Delta} G(w) = 1$. $\square$

²⁶Otherwise, since $Q \in \mathbb{R}_+^n$, we would have that $Q = 0$ and, in particular, $G(Q) = 0$, a contradiction.

Theorem 4. Let $K : \mathbb{R}_{++}^n \rightarrow \mathbb{R}_{++}$. The following statements are equivalent:

- (i) $K \in \mathcal{K}_{\text{nor}}$;
- (ii) K is monotone, continuous, quasiconcave, positive homogeneous, and such that $K(e) = 1$;
- (iii) There exists an upper semicontinuous and concave function $\Gamma : \Delta \rightarrow [0, \infty)$ such that $\max_{w \in \Delta} \Gamma(w) = 1$ and

$$K(P) = \min_{w \in \Delta} \frac{\langle w, P \rangle}{\Gamma(w)} \quad \forall P \in \mathbb{R}_{++}^n. \quad (172)$$

- (iv) There exists $G^* \in \mathcal{G}_{\text{nor-con}}$ such that $K = K_{G^*}$.
- (v) There exists $G^* \in \mathcal{G}_{\text{nor-con}}$ such that

$$K(P) = \min_{w \in \Delta} \frac{\langle w, P \rangle}{G^*(w)} \quad \forall P \in \mathbb{R}_{++}^n.$$

Moreover, we have that:

1. $\Gamma : \Delta \rightarrow [0, \infty)$ is such that

$$\Gamma(w) = \frac{1}{\sup \{K(P) : P \in \mathbb{R}_{++}^n \text{ and } \langle w, P \rangle \leq 1\}} \quad \forall w \in \Delta$$

and it is the unique upper semicontinuous and concave function $\tilde{\Gamma}$ satisfying (172) and such that $\max_{w \in \Delta} \tilde{\Gamma}(w) = 1$.

2. G^* is the unique element in $\mathcal{G}_{\text{nor-con}}$ such that $K = K_{G^*}$.
3. $\Gamma(w) = G^*(w)$ for all $w \in \Delta$. In particular, Γ is continuous and $G^*(0) = 0$ and $G^*(Q) = \|Q\|_1 \Gamma(Q/\|Q\|_1)$ for all $Q \in \mathbb{R}_+^n \setminus \{0\}$.
4. If $G \in \mathcal{G}_{\text{nor}}$ and $K = K_G$, then $G^* = \min \left\{ \hat{G} : \hat{G} \in \mathcal{G}_{\text{nor-con}} \text{ and } \hat{G} \geq G \right\}$.

Proof. (i) implies (ii). By definition of $\mathcal{K}_{\text{nor}}$, it is obvious.

(ii) implies (iii) and point 1. Call $\bar{K}$ the extension of K to $\mathbb{R}^n$ such that $\bar{K}(P) = 0$ for all $P \notin \mathbb{R}_{++}^n$. Since $K > 0$ and K is monotone and positive homogeneous, it is immediate to see that $\bar{K}$ is monotone and positive homogeneous. Since $\mathbb{R}_{++}^n$ is open and K is continuous on $\mathbb{R}_{++}^n$ and $K > 0$, we have that

$$\{\bar{K} > t\} = \{P \in \mathbb{R}_{++}^n : K(P) > t\} \quad \forall t \geq 0.$$

It follows that the latter set is relatively open in $\mathbb{R}_{++}^n$, proving that the latter and the former are open. Moreover, since $\bar{K} \geq 0$, we have that $\{\bar{K} > t\} = \mathbb{R}^n$ for all $t < 0$, proving that $\bar{K}$ is lower semicontinuous. Since K is quasiconcave on $\mathbb{R}_{++}^n$ and $K > 0$, we have that

$$\{\bar{K} \geq t\} = \{P \in \mathbb{R}_{++}^n : K(P) \geq t\} \quad \forall t > 0.$$

It follows that the first set is convex. Moreover, since $\bar{K} \geq 0$, we have that $\{\bar{K} \geq t\} = \mathbb{R}^n$ for all $t \leq 0$, proving that $\bar{K}$ is quasiconcave. By (Cerreia-Vioglio et al., 2011, Theorem 8.1 and

Proposition 8.1 and their proofs) and since $\bar{K}$ is monotone, lower semicontinuous, quasiconcave, positive homogeneous, and such that $\bar{K}(e) = K(e) = 1$, we have that

$$\bar{K}(P) = \min_{w \in \tilde{\Delta}} \left\{ \frac{\langle w, P \rangle^+}{c_1(w)} - \frac{\langle w, P \rangle^-}{c_2(w)} \right\} \quad \forall P \in \mathbb{R}^n \quad (173)$$

where $\tilde{\Delta}$ is a nonempty, closed, and convex subset of Δ , $c_1 : \tilde{\Delta} \rightarrow [0, \infty)$ is upper semicontinuous and concave, $c_2 : \tilde{\Delta} \rightarrow (0, \infty]$ is lower semicontinuous and convex. Moreover, $\tilde{\Delta} = \{w \in \Delta : \text{dom } \bar{H}(\cdot, w) \neq \emptyset\}$ where $\bar{H} : \mathbb{R} \times \Delta \rightarrow (-\infty, \infty]$ is defined by

$$\bar{H}(t, w) = \sup \{ \bar{K}(P) : \langle w, P \rangle \leq t \} \quad \forall (t, w) \in \mathbb{R} \times \Delta$$

and c_1 and c_2 are unique with $\bar{H}(1, w) = \frac{1}{c_1(w)}$ for all $w \in \tilde{\Delta}$. Since $\bar{K}(P) = 0$ for all $P \notin \mathbb{R}_{++}^n$, it is immediate to see that $\bar{H}(t, w) = 0$ for all $w \in \Delta$ and for all $t \leq 0$, that is, $(-\infty, 0] \subseteq \text{dom } \bar{H}(\cdot, w)$ for all $w \in \Delta$, proving that $\tilde{\Delta} = \Delta$. In particular, this implies that

$$\begin{aligned} c_1(w) &= \frac{1}{\bar{H}(1, w)} = \frac{1}{\sup \{ \bar{K}(P) : \langle w, P \rangle \leq 1 \}} \\ &= \frac{1}{\sup \{ K(P) : P \in \mathbb{R}_{++}^n \text{ and } \langle w, P \rangle \leq 1 \}} \quad \forall w \in \Delta. \end{aligned}$$

By (173) and since $\bar{K}(e) = 1$ and $\bar{K}(-e) = 0$, we have that $1 = \min_{w \in \Delta} \frac{1}{c_1(w)} = \frac{1}{\max_{w \in \Delta} c_1(w)}$ and $0 = \min_{w \in \Delta} -\frac{1}{c_2(w)} = -\frac{1}{\min_{w \in \Delta} c_2(w)}$, that is, $\max_{w \in \Delta} c_1(w) = 1$ and $\min_{w \in \Delta} c_2(w) = \infty$. By restricting (173) to $\mathbb{R}_{++}^n$ and setting $\Gamma = c_1$, Equation (172) follows. Next, consider $\tilde{\Gamma} : \Delta \rightarrow [0, \infty)$ such that $\tilde{\Gamma}$ is upper semicontinuous, concave, and such that $\max_{w \in \Delta} \tilde{\Gamma}(w) = 1$ and $\tilde{\Gamma}$ satisfies (172). By the uniqueness part of (Cerrea-Vioglio et al., 2011, Proposition 8.1) and since $c_2 = \infty$ and $\bar{K}(P) = 0$ for all $P \notin \mathbb{R}_{++}^n$, this implies that

$$\bar{K}(P) = \min_{w \in \Delta} \left\{ \frac{\langle w, P \rangle^+}{\tilde{\Gamma}(w)} - \frac{\langle w, P \rangle^-}{c_2(w)} \right\} \quad \forall P \in \mathbb{R}^n,$$

proving that $\Gamma = c_1 = \tilde{\Gamma}$ and uniqueness.

(iii) implies (iv). Define $G^* : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ by $G^*(0) = 0$ and $G^*(Q) = \|Q\|_1 \Gamma(Q/\|Q\|_1)$ for all $Q \in \mathbb{R}_+^n \setminus \{0\}$. Note that $G^*|_{\Delta} = \Gamma$. Consider $Q, \tilde{Q} \in \mathbb{R}_+^n$ such that $Q \geq \tilde{Q}$. Since $\Gamma \geq 0$, if $\tilde{Q} = 0$, then $G^*(Q) \geq 0 = G^*(\tilde{Q})$. If $\tilde{Q} > 0$, then $Q > 0$. In this case, if $Q = \tilde{Q}$, then $G^*(Q) \geq G^*(\tilde{Q})$. Assume $Q > \tilde{Q}$. Define $\delta = Q - \tilde{Q} \in \mathbb{R}_+^n$. Since $\tilde{Q}, \delta > 0$, Γ is concave, $\| \cdot \|_1$

is additive on $\mathbb{R}_+^n$, and $G^\star \geq 0$, note that

$$\begin{aligned}
G^\star(Q) &= \|Q\|_1 \Gamma(Q/\|Q\|_1) = \|\tilde{Q} + \delta\|_1 \Gamma\left(\frac{\tilde{Q} + \delta}{\|\tilde{Q} + \delta\|_1}\right) \\
&= \|\tilde{Q} + \delta\|_1 \Gamma\left(\frac{\|\tilde{Q}\|_1}{\|\tilde{Q} + \delta\|_1} \frac{\tilde{Q}}{\|\tilde{Q}\|_1} + \frac{\|\delta\|_1}{\|\tilde{Q} + \delta\|_1} \frac{\delta}{\|\delta\|_1}\right) \\
&\geq \|\tilde{Q} + \delta\|_1 \frac{\|\tilde{Q}\|_1}{\|\tilde{Q} + \delta\|_1} \Gamma\left(\frac{\tilde{Q}}{\|\tilde{Q}\|_1}\right) + \|\tilde{Q} + \delta\|_1 \frac{\|\delta\|_1}{\|\tilde{Q} + \delta\|_1} \Gamma\left(\frac{\delta}{\|\delta\|_1}\right) \\
&= G^\star(\tilde{Q}) + G^\star(\delta) \geq G^\star(\tilde{Q}),
\end{aligned}$$

proving that $G^\star$ is monotone. Consider $\{Q^k\}_{k \in \mathbb{N}} \subseteq \mathbb{R}_+^n$ such that $Q^k \rightarrow Q$. Note that there exists a subsequence $\{Q^{k_m}\}_{m \in \mathbb{N}} \subseteq \{Q^k\}_{k \in \mathbb{N}} \subseteq \mathbb{R}_+^n$ such that $\limsup_k G^\star(Q^k) = \lim_m G^\star(Q^{k_m})$. If there exists a further subsequence $\{Q^{k_{m(l)}}\}_{l \in \mathbb{N}} \subseteq \{Q^{k_m}\}_{m \in \mathbb{N}} \subseteq \{Q^k\}_{k \in \mathbb{N}} \subseteq \mathbb{R}_+^n$ such that $Q^{k_{m(l)}} = 0$ for all $l \in \mathbb{N}$, then $Q = 0$ and $\limsup_k G^\star(Q^k) = \lim_m G^\star(Q^{k_m}) = \lim_l G^\star(Q^{k_{m(l)}}) = 0 = G^\star(Q)$. If this further subsequence does not exist, there exists $\bar{m} \in \mathbb{N}$ such that $Q^{k_m} > 0$ for all $m \geq \bar{m}$, therefore we can assume without loss of generality that $Q^{k_m} \neq 0$ for all $m \in \mathbb{N}$ and define $w^{k_m} = Q^{k_m} / \|Q^{k_m}\|_1 \in \Delta$ for all $m \in \mathbb{N}$. Since Δ is compact, there exists a further subsequence $\{Q^{k_{m(l)}}\}_{l \in \mathbb{N}} \subseteq \{Q^{k_m}\}_{m \in \mathbb{N}} \subseteq \{Q^k\}_{k \in \mathbb{N}} \subseteq \mathbb{R}_+^n$ such that $\lim_l w^{k_{m(l)}} = w \in \Delta$. Since Γ is upper semicontinuous and $\Gamma \geq 0$, we can conclude that

$$\limsup_k G^\star(Q^k) = \lim_m G^\star(Q^{k_m}) = \lim_l G^\star(Q^{k_{m(l)}}) = \limsup_l \|Q^{k_{m(l)}}\|_1 \Gamma(w^{k_{m(l)}}) \leq \|Q\|_1 \Gamma(w).$$

If $Q = 0$, then $\|Q\|_1 = 0$ and $\|Q\|_1 \Gamma(w) = 0 = G^\star(Q)$. If $Q > 0$, then $w = Q / \|Q\|_1$ and $\|Q\|_1 \Gamma(w) = G^\star(Q)$, proving upper semicontinuity. Fix $Q \in \mathbb{R}_+^n$ and $\lambda \geq 0$. If either $Q = 0$ or $\lambda = 0$, then $G^\star(\lambda Q) = G^\star(0) = 0 = \lambda G^\star(Q)$. Otherwise, we have that

$$G^\star(\lambda Q) = \|\lambda Q\|_1 \Gamma(\lambda Q / \|\lambda Q\|_1) = \lambda \|Q\|_1 \Gamma(Q / \|Q\|_1) = \lambda G^\star(Q),$$

proving that $G^\star$ is positive homogeneous. Since $G^\star|_\Delta = \Gamma$ and $\max_{w \in \Delta} \Gamma(w) = 1$, we have that $\max_{w \in \Delta} G^\star(w) = 1$. Next, fix $Q, \tilde{Q} \in \mathbb{R}_+^n$ and $\lambda \in (0, 1)$. If either $Q = 0$ or $\tilde{Q} = 0$, then either $G^\star(\lambda Q + (1 - \lambda)\tilde{Q}) = G^\star((1 - \lambda)\tilde{Q}) = (1 - \lambda)G^\star(\tilde{Q}) = \lambda G^\star(Q) + (1 - \lambda)G^\star(\tilde{Q})$ or $G^\star(\lambda Q + (1 - \lambda)\tilde{Q}) = G^\star(\lambda Q) = \lambda G^\star(Q) = \lambda G^\star(Q) + (1 - \lambda)G^\star(\tilde{Q})$. Since Γ is concave

and $\|\cdot\|_1$ is additive and positive homogeneous on $\mathbb{R}_+^n$, if instead $Q, \tilde{Q} > 0$, note that

$$\begin{aligned}
& G^* \left(\lambda Q + (1 - \lambda) \tilde{Q} \right) \\
&= \left\| \lambda Q + (1 - \lambda) \tilde{Q} \right\|_1 \Gamma \left(\frac{\lambda Q + (1 - \lambda) \tilde{Q}}{\left\| \lambda Q + (1 - \lambda) \tilde{Q} \right\|_1} \right) \\
&= \left\| \lambda Q + (1 - \lambda) \tilde{Q} \right\|_1 \Gamma \left(\frac{\lambda \|Q\|_1}{\left\| \lambda Q + (1 - \lambda) \tilde{Q} \right\|_1} \frac{Q}{\|Q\|_1} + \frac{(1 - \lambda) \|\tilde{Q}\|_1}{\left\| \lambda Q + (1 - \lambda) \tilde{Q} \right\|_1} \frac{\tilde{Q}}{\|\tilde{Q}\|_1} \right) \\
&\geq \left\| \lambda Q + (1 - \lambda) \tilde{Q} \right\|_1 \frac{\lambda \|Q\|_1}{\left\| \lambda Q + (1 - \lambda) \tilde{Q} \right\|_1} \Gamma \left(\frac{Q}{\|Q\|_1} \right) + \left\| \lambda Q + (1 - \lambda) \tilde{Q} \right\|_1 \frac{(1 - \lambda) \|\tilde{Q}\|_1}{\left\| \lambda Q + (1 - \lambda) \tilde{Q} \right\|_1} \Gamma \left(\frac{\tilde{Q}}{\|\tilde{Q}\|_1} \right) \\
&= \lambda \|Q\|_1 \Gamma \left(\frac{Q}{\|Q\|_1} \right) + (1 - \lambda) \|\tilde{Q}\|_1 \Gamma \left(\frac{\tilde{Q}}{\|\tilde{Q}\|_1} \right) \\
&= \lambda G^*(Q) + (1 - \lambda) G^*(\tilde{Q}),
\end{aligned}$$

proving concavity and $G^* \in \mathcal{G}_{\text{nor-con}}$. By Lemma 20 and since $G_{|\Delta}^* = \Gamma$, we can conclude that

$$K(P) = \min_{w \in \Delta} \frac{\langle w, P \rangle}{\Gamma(w)} = \min_{w \in \Delta} \frac{\langle w, P \rangle}{G^*(w)} = K_{G^*}(P) \quad \forall P \in \mathbb{R}_{++}^n,$$

proving the implication.

(iv) implies (v). By Lemma 20, the implication follows.

(v) implies (i). Since K is the inf of monotone, continuous, affine, and positive homogeneous functionals, we have that K is monotone, upper semicontinuous, concave, and positive homogeneous. By (Borwein and Lewis, 2006, Theorem 4.1.3) and since K is concave, it follows that K is continuous. Finally, by Lemma 20 and since $\max_{w \in \Delta} G^*(w) = 1$, we have that $K(e) = 1$, proving the implication.

3. By points (iii) and (v) and point 1 and since G^* is continuous and concave and such that $\max_{w \in \Delta} G^*(w) = 1$, we have that

$$K(P) = \min_{w \in \Delta} \frac{\langle w, P \rangle}{\Gamma(w)} = \min_{w \in \Delta} \frac{\langle w, P \rangle}{G^*(w)} \quad \forall P \in \mathbb{R}_{++}^n,$$

proving that $\Gamma(w) = G^*(w)$ for all $w \in \Delta$. Since G^* is continuous and positive homogeneous, the rest of the point follows.

2. Consider any other function $G \in \mathcal{G}_{\text{nor-con}}$ satisfying either point (iv) or (v). By the same argument above and the equivalence with (iii), we have that $G(w) = \Gamma(w) = G^*(w)$ for all $w \in \Delta$. By positive homogeneity, we can conclude that $G(Q) = G^*(Q)$ for all $Q \in \mathbb{R}_+^n \setminus \{0\}$. By continuity, we have that $G(0) = 0 = G^*(0)$, proving uniqueness of G^* .

4. Define $\bar{K} : \mathbb{R}^n \rightarrow \mathbb{R}$ by $\bar{K}(P) = 0$ for all $P \notin \mathbb{R}_{++}^n$ and $\bar{K}(P) = K_G(P)$ for all $P \in \mathbb{R}_{++}^n$.

Define $\hat{H} : \mathbb{R} \times \Delta \rightarrow (-\infty, \infty]$ by

$$\hat{H}(t, w) = \begin{cases} \frac{t}{G(w)} & t > 0 \\ 0 & t \leq 0 \end{cases} \quad \forall (t, w) \in \mathbb{R} \times \Delta.$$

By Lemma 20, it is immediate to see that

$$\bar{K}(P) = \min_{w \in \Delta} \hat{H}(\langle w, P \rangle, w) \quad \forall P \in \mathbb{R}^n.$$

By points 1 and 3 and definition of $\bar{K}$, we have that

$$\begin{aligned} \frac{1}{G^*(\tilde{w})} &= \sup \{ K(P) : P \in \mathbb{R}_{++}^n \text{ and } \langle \tilde{w}, P \rangle \leq 1 \} = \sup \{ \bar{K}(P) : \langle \tilde{w}, P \rangle \leq 1 \} \\ &= \sup_{P \in \mathbb{R}^n : \langle \tilde{w}, P \rangle \leq 1} \inf_{w \in \Delta} \hat{H}(\langle w, P \rangle, w) \leq \inf_{w \in \Delta} \sup_{P \in \mathbb{R}^n : \langle \tilde{w}, P \rangle \leq 1} \hat{H}(\langle w, P \rangle, w) = \frac{1}{G(\tilde{w})} \quad \forall \tilde{w} \in \Delta, \end{aligned}$$

proving that $G^* \geq G$ on Δ . Since G and G^* are continuous and positive homogeneous, it follows that $G^* \geq G$. Next, consider $\tilde{G} \in \mathcal{G}_{\text{nor-con}}$ and $\tilde{G} \geq G$. It follows that $K_{\tilde{G}} \leq K_G = K_{G^*}$. By points 1 and 3, we have that

$$\begin{aligned} \frac{1}{G^*(w)} &= \sup \{ K_{G^*}(P) : P \in \mathbb{R}_{++}^n \text{ and } \langle w, P \rangle \leq 1 \} \\ &\geq \sup \{ K_{\tilde{G}}(P) : P \in \mathbb{R}_{++}^n \text{ and } \langle w, P \rangle \leq 1 \} \\ &= \frac{1}{\tilde{G}(w)} \quad \forall w \in \Delta, \end{aligned}$$

proving that $\tilde{G} \geq G^*$ on Δ . Since $\tilde{G}$ and G^* are continuous and positive homogeneous, it follows that $\tilde{G} \geq G^*$, proving the point. $\square$

F.4 Fragile Sectors

By Lemma 20, if $G \in \mathcal{G}_{\text{nor}}$, then

$$K_G(P) = \min_{w \in \Delta} \frac{\langle w, P \rangle}{G(w)} \quad \forall P \in \mathbb{R}_{++}^n$$

and

$$\kappa_G(p) = \min_{w \in \Delta} \{ M(w, p) + c(w) \} \quad \forall p \in \mathbb{R}^n \quad (174)$$

where $M : \Delta \times \mathbb{R}^n \rightarrow \mathbb{R}$ is defined by $M(w, p) = \ln \left(\sum_{j=1}^n e^{p_j} w_j \right)$ for all $(w, p) \in \Delta \times \mathbb{R}^n$ and $c : \Delta \rightarrow [0, \infty]$ is defined by $c(w) = -\ln G(w)$ for all $w \in \Delta$. Since $K_G \in \mathcal{K}_{\text{nor}}$, we have that κ_G is normalized, monotone, and translation invariant. We define $R(\cdot || \cdot) : \Delta \times \Delta \rightarrow [0, \infty]$ by

$$R(w || \tilde{w}) = \begin{cases} \sum_{j=1}^n \ln \left(\frac{w_j}{\tilde{w}_j} \right) w_j & w \ll \tilde{w} \\ \infty & \text{otherwise} \end{cases} \quad \forall (w, \tilde{w}) \in \Delta \times \Delta.$$

We define the absolute entropy $\text{Ent} : \Delta \rightarrow \mathbb{R}$ by $\text{Ent}(w) = -\sum_{j=1}^n w_j \ln w_j$ for all $w \in \Delta$. Recall that Ent is continuous and strictly concave.²⁷ Given a convex subset C of Δ , we will say that a function $f : C \rightarrow \mathbb{R} \cup \{\infty\}$ is *entropy convex* if and only if $f + \text{Ent}$ is convex. By subtracting Ent , it is immediate to see that if f is entropy convex, then f is strictly convex on its effective domain and convex in general. Given a function $A : C \rightarrow [1, \infty]$, we will say it is *entropy log-convex* if and only if $d : C \rightarrow [0, \infty]$, defined by $d(w) = \ln A(w)$ for all $w \in C$, is entropy convex.

Lemma 21. *Let $G \in \mathcal{G}_{\text{nor}}$. If $\kappa_G : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex, then there exists a grounded, lower semicontinuous, and convex $d : \Delta \rightarrow [0, \infty]$ such that*

$$\kappa_G(p) = \max_{w \in \Delta} \{\langle w, p \rangle - d(w)\} \quad \forall p \in \mathbb{R}^n \quad (175)$$

and

$$d(w) = \sup_{\tilde{w} \in \text{dom } c} \{R(w||\tilde{w}) - c(\tilde{w})\} \quad \forall w \in \Delta. \quad (176)$$

Moreover, if $\tilde{w} \in \text{int } \Delta$, then $w \mapsto d(w) - R(w||\tilde{w})$ is lower semicontinuous and convex on Δ . In particular, d is lower semicontinuous and entropy convex.

Proof. Define $\bar{\kappa}_G : \mathbb{R}^n \rightarrow \mathbb{R}$ by $\bar{\kappa}_G(p) = -\kappa_G(-p)$ for all $p \in \mathbb{R}^n$. We have that $\bar{\kappa}_G$ is normalized, monotone, translation invariant, concave, and such that

$$\bar{\kappa}_G(p) = \max_{w \in \Delta} \{m(w, p) - c(w)\} \quad \forall p \in \mathbb{R}^n \quad (177)$$

where $m : \Delta \times \mathbb{R}^n \rightarrow \mathbb{R}$ is defined by $m(w, p) = -\ln \left(\sum_{j=1}^n e^{-p_j} w_j \right)$ for all $(w, p) \in \Delta \times \mathbb{R}^n$. By [Cerrei-Vioglio et al. \(2014\)](#) and since $\bar{\kappa}_G$ is normalized, monotone, translation invariant, and concave, we have that $d : \Delta \rightarrow [0, \infty]$, defined by $d(w) = \sup_{p \in \mathbb{R}^n} \{\bar{\kappa}_G(p) - \langle w, p \rangle\}$ for all $w \in \Delta$, is such that

$$\bar{\kappa}_G(p) = \min_{w \in \Delta} \{\langle w, p \rangle + d(w)\} \quad \forall p \in \mathbb{R}^n.$$

By definition of d and since $\bar{\kappa}_G$ is normalized, we have that d is grounded, lower semicontinuous, and convex. By definition of d and (177) and Donsker-Varadhan's variational formula, we also have that

$$\begin{aligned} d(w) &= \sup_{p \in \mathbb{R}^n} \left\{ \max_{\tilde{w} \in \Delta} \{m(\tilde{w}, p) - c(\tilde{w})\} - \langle w, p \rangle \right\} = \sup_{p \in \mathbb{R}^n} \sup_{\tilde{w} \in \Delta} \{m(\tilde{w}, p) - c(\tilde{w}) - \langle w, p \rangle\} \\ &= \sup_{p \in \mathbb{R}^n} \sup_{\tilde{w} \in \text{dom } c} \{m(\tilde{w}, p) - c(\tilde{w}) - \langle w, p \rangle\} = \sup_{\tilde{w} \in \text{dom } c} \sup_{p \in \mathbb{R}^n} \{m(\tilde{w}, p) - \langle w, p \rangle - c(\tilde{w})\} \\ &= \sup_{\tilde{w} \in \text{dom } c} \left\{ \sup_{p \in \mathbb{R}^n} \{m(\tilde{w}, p) - \langle w, p \rangle\} - c(\tilde{w}) \right\} = \sup_{\tilde{w} \in \text{dom } c} \{R(w||\tilde{w}) - c(\tilde{w})\} \quad \forall w \in \Delta. \end{aligned}$$

Since $\kappa_G(p) = -\bar{\kappa}_G(-p)$ for all $p \in \mathbb{R}^n$, the main statement and (176) follow. Fix $\check{w} \in \text{int } \Delta$. Since $\check{w} \in \text{int } \Delta$, we have that $R(w||\check{w}) < \infty$ for all $w \in \Delta$. For each $\tilde{w} \in \text{dom } c$ define $f_{\tilde{w}} : \Delta \rightarrow \mathbb{R} \cup \{\infty\}$ by $f_{\tilde{w}}(w) = R(w||\tilde{w}) - c(\tilde{w}) - R(w||\check{w})$ for all $w \in \Delta$. Fix $\tilde{w} \in \text{dom } c$. Since

²⁷Define $\phi : [0, \infty) \rightarrow \mathbb{R}$ by $\phi(0) = 0$ and $\phi(t) = t \ln(t)$ for all $t > 0$. Observe that ϕ is continuous and strictly convex.

$R(w||\check{w}) \in \mathbb{R}$ for all $w \in \Delta$ and $c(\tilde{w}) \in \mathbb{R}$, note that $w \in \text{dom } f_{\tilde{w}}$ if and only if $f_{\tilde{w}}(w) < \infty$ if and only if $R(w||\tilde{w}) < \infty$, proving that $\text{dom } f_{\tilde{w}} = \text{dom } R(\cdot||\tilde{w})$ is convex. Since $R(w||\check{w}) \in \mathbb{R}$ for all $w \in \Delta$ and $R(w||\tilde{w}) \in \mathbb{R}$ for all $w \in \text{dom } f_{\tilde{w}}$, we have that $w_j = 0$ whenever $\tilde{w}_j = 0$ and

$$\begin{aligned} f_{\tilde{w}}(w) &= R(w||\tilde{w}) - c(\tilde{w}) - R(w||\check{w}) = \sum_{j=1}^n \ln \left(\frac{w_j}{\tilde{w}_j} \right) w_j - c(\tilde{w}) - \sum_{j=1}^n \ln \left(\frac{w_j}{\check{w}_j} \right) w_j \\ &= \sum_{j=1}^n \ln \left(\frac{\check{w}_j}{\tilde{w}_j} \right) w_j - c(\tilde{w}) \quad \forall w \in \text{dom } f_{\tilde{w}}. \end{aligned}$$

This implies that $f_{\tilde{w}}$ is affine over $\text{dom } f_{\tilde{w}}$ and, in particular, convex on Δ . By definition of d and since $\tilde{w}$ was arbitrarily chosen, we have that

$$\begin{aligned} d(w) - R(w||\check{w}) &= \sup_{\tilde{w} \in \text{dom } c} \{R(w||\tilde{w}) - c(\tilde{w})\} - R(w||\check{w}) \\ &= \sup_{\tilde{w} \in \text{dom } c} \{R(w||\tilde{w}) - c(\tilde{w}) - R(w||\check{w})\} = \sup_{\tilde{w} \in \text{dom } c} f_{\tilde{w}}(w) \quad \forall w \in \Delta, \end{aligned}$$

proving that $w \mapsto d(w) - R(w||\check{w})$ is convex. Since $\check{w} \in \text{int } \Delta$ and d is lower semicontinuous, we have that $w \mapsto R(w||\check{w})$ is continuous and $w \mapsto d(w) - R(w||\check{w})$ is lower semicontinuous. Fix $\check{w} \in \text{int } \Delta$. Since $w \mapsto \sum_{j=1}^n \ln(\check{w}_j) w_j$ is affine and $w \mapsto d(w) - R(w||\check{w})$ is convex, we have that $w \mapsto d(w) + \text{Ent}(w) = d(w) - R(w||\check{w}) - \sum_{j=1}^n \ln(\check{w}_j) w_j$, proving that $d + \text{Ent}$ is convex. $\square$

Proposition 12. *Let $K : \mathbb{R}_{++}^n \rightarrow \mathbb{R}_{++}$. The following statements are equivalent:*

- (i) $K \in \mathcal{K}_{\text{nor}}$ and κ is convex;
- (ii) There exists $G \in \mathcal{G}_{\text{nor}}$ such that $K = K_G$ and $\kappa = \kappa_G$ is convex;
- (iii) There exists a unique $G^* \in \mathcal{G}_{\text{nor-con}}$ such that $K = K_{G^*}$, $\kappa = \kappa_{G^*}$, and

$$G^*(Q) = \min_{w \in C} A(w) \Pi_{j=1}^n \left(\frac{Q_j}{w_j} \right)^{w_j} \quad \forall Q \in \mathbb{R}_+^n$$

where C is a nonempty, compact, and convex subset of Δ and $A : C \rightarrow [1, \infty]$ is lower semicontinuous, entropy log-convex, and such that $\min_{w \in C} A(w) = 1$.

Proof. (i) implies (ii). By Theorem 4, the implication follows.

(ii) implies (iii). Define $L : \mathbb{R}_{++}^n \rightarrow \mathbb{R}^n$ by $L(P)$ being the vector whose j -th component is $\ln(P_j)$ for all $j \in [n]$ and for all $P \in \mathbb{R}_{++}^n$. By Lemma 21 and definition of κ_G , this implies that

$$K(P) = \max_{w \in \Delta} e^{\langle w, L(P) \rangle - d(w)} = \max_{w \in \Delta} \frac{e^{\langle w, L(P) \rangle}}{e^{d(w)}} = \max_{w \in \Delta} \frac{\Pi_{j=1}^n P_j^{w_j}}{e^{d(w)}} \quad \forall P \in \mathbb{R}_{++}^n \quad (178)$$

where $d : \Delta \rightarrow [0, \infty]$ is grounded, lower semicontinuous, and convex. For each $w \in \Delta$ define $K_w : \mathbb{R}_{++}^n \rightarrow \mathbb{R}_+$ by $K_w(P) = \Pi_{j=1}^n P_j^{w_j} / e^{d(w)}$ for all $P \in \mathbb{R}_{++}^n$. If $d(w) = \infty$, note that $K_w = 0$. Given $w \in \Delta$, define $H, H_w : \Delta \rightarrow [0, \infty]$ by

$$H(\tilde{w}) = \sup \{K(P) : P \in \mathbb{R}_{++}^n \text{ s.t. } \langle \tilde{w}, P \rangle \leq 1\} \quad \forall \tilde{w} \in \Delta$$

and

$$H_w(\tilde{w}) = \sup \{ K_w(P) : P \in \mathbb{R}_{++}^n \text{ s.t. } \langle \tilde{w}, P \rangle \leq 1 \} \quad \forall \tilde{w} \in \Delta.$$

Note that

$$H_w(\tilde{w}) = \frac{e^{R(w)\|\tilde{w}\|}}{e^{d(w)}} = \frac{1}{e^{d(w)-R(w)\|\tilde{w}\|}} = \frac{1}{e^{d(w)\prod_{j=1}^n \left(\frac{\tilde{w}_j}{w_j}\right)^{w_j}}} > 0 \quad \forall (\tilde{w}, w) \in \Delta \times \text{dom } d$$

while $H_w(\tilde{w}) = 0$ for all $(\tilde{w}, w) \in \Delta \times (\text{dom } d)^c$. By (178), note that $H(\tilde{w}) = \sup_{w \in \Delta} H_w(\tilde{w}) = \sup_{w \in \text{dom } d} H_w(\tilde{w})$ for all $\tilde{w} \in \Delta$ and

$$H(\tilde{w}) = \sup_{w \in \text{dom } d} H_w(\tilde{w}) = \sup_{w \in \text{dom } d} \left\{ \frac{1}{e^{d(w)\prod_{j=1}^n \left(\frac{\tilde{w}_j}{w_j}\right)^{w_j}}} \right\} = \frac{1}{\inf_{w \in \text{dom } d} e^{d(w)\prod_{j=1}^n \left(\frac{\tilde{w}_j}{w_j}\right)^{w_j}}} \quad \forall \tilde{w} \in \Delta.$$

By Lemma 15 and Theorem 4 and points 1–3 and since $K = K_G \in \mathcal{K}_{\text{nor}}$, we can conclude that $K = K_{G^*}$ where $\Gamma : \Delta \rightarrow [0, \infty)$ and $G^* : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ are such that

$$\Gamma(\tilde{w}) = \frac{1}{H(\tilde{w})} = \inf_{w \in \text{dom } d} e^{d(w)\prod_{j=1}^n \left(\frac{\tilde{w}_j}{w_j}\right)^{w_j}} \quad \forall \tilde{w} \in \Delta$$

and

$$G^*(Q) = \inf_{w \in \text{dom } d} e^{d(w)\prod_{j=1}^n \left(\frac{Q_j}{w_j}\right)^{w_j}} \quad \forall Q \in \mathbb{R}_+^n.$$

By Lemma 21, we have that $C = \text{cl}(\text{dom } d)$ is a nonempty, compact, and convex subset of Δ and $A = e^d$ from C to $[1, \infty]$ is lower semicontinuous, entropy log-convex, and such that $\min_{w \in C} A(w) = 1$. We next show that the inf can be extended from $\text{dom } d$ to C and it is a min. Consider $Q \in \mathbb{R}_+^n$. We have two cases:

1. $A(w)\prod_{j=1}^n \left(\frac{Q_j}{w_j}\right)^{w_j} > 0$ for all $w \in \text{dom } d$. In this case, since $A(w) \in [1, \infty)$, we have that for each $w \in \text{dom } d$

$$w_j > 0 \implies Q_j > 0.$$

Define $\Delta(Q) = \{w \in \Delta : Q_j = 0 \implies w_j = 0\}$. Clearly, $\Delta(Q)$ is closed and $\text{dom } d \subseteq \Delta(Q)$ and, in particular, $C \subseteq \Delta(Q)$. Note also that $\prod_{j=1}^n \left(\frac{Q_j}{w_j}\right)^{w_j} = \exp\left(\sum_{j=1}^n \ln \frac{Q_j}{w_j} w_j\right) = \exp\left(\sum_{j=1}^n \ln Q_j w_j + \text{Ent}(w)\right)$ for all $w \in \Delta(Q)$. Note that the function $w \mapsto \sum_{j=1}^n \ln Q_j w_j + \text{Ent}(w)$ is real-valued and continuous on $\Delta(Q)$ and, in particular, on C . This implies that the function

$$w \mapsto A(w)\prod_{j=1}^n \left(\frac{Q_j}{w_j}\right)^{w_j} = \exp\left(d(w) + \sum_{j=1}^n \ln Q_j w_j + \text{Ent}(w)\right)$$

is lower semicontinuous on C , proving that the inf over C is a min. Moreover, since

$$A(w)\prod_{j=1}^n \left(\frac{Q_j}{w_j}\right)^{w_j} = \exp\left(d(w) + \sum_{j=1}^n \ln Q_j w_j + \text{Ent}(w)\right) = \infty \quad \forall w \in C \setminus \text{dom } d,$$

we have that extending the inf to C does not change the value of the minimization.

2. $A(\bar{w}) \prod_{j=1}^n \left(\frac{Q_j}{\bar{w}_j} \right)^{\bar{w}_j} = 0$ for some $\bar{w} \in \text{dom } d$. Since $w \mapsto A(w) \prod_{j=1}^n \left(\frac{Q_j}{w_j} \right)^{w_j} \geq 0$ for all $w \in C$, we have that

$$\begin{aligned} G^*(Q) &= \inf_{w \in \text{dom } d} e^{d(w)} \prod_{j=1}^n \left(\frac{Q_j}{w_j} \right)^{w_j} = \inf_{w \in C} e^{d(w)} \prod_{j=1}^n \left(\frac{Q_j}{w_j} \right)^{w_j} \\ &= A(\bar{w}) \prod_{j=1}^n \left(\frac{Q_j}{\bar{w}_j} \right)^{\bar{w}_j} = 0, \end{aligned}$$

proving that inf can be extended from $\text{dom } d$ to C and it is a min.

Points 1 and 2 prove the implication.

(iii) implies (i). Define $\tilde{C} = \{w \in C : A(w) < \infty\}$. Since $\tilde{C}$ is a nonempty subset of a compact set and A is such that $\min_{w \in C} A(w) = 1$ and entropy log-convex and, in particular, quasiconvex, we have that $\tilde{C}$ is nonempty, precompact, and convex. Define $\mathfrak{G} : \tilde{C} \times \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ by

$$\mathfrak{G}(w, Q) = A(w) \prod_{j=1}^n \left(\frac{Q_j}{w_j} \right)^{w_j} \quad \forall (w, Q) \in \tilde{C} \times \mathbb{R}_+^n.$$

Clearly, $\mathfrak{G}(w, \cdot) \in \mathcal{G}_{\text{con}}$ for all $w \in \tilde{C}$. Define $d : \tilde{C} \rightarrow [0, \infty]$ by $d(w) = \ln A(w)$ for all $w \in \tilde{C}$. Given $Q \in \mathbb{R}_{++}^n$, note that $\mathfrak{G}(w, Q) = \|Q\|_1 e^{d(w) + \text{Ent}(w) + \sum_{j=1}^n \ln(\check{w}_j) w_j}$ where $\check{w} = Q / \|Q\|_1$. Since A is lower semicontinuous on C and entropy log-convex, it follows that $w \mapsto d(w) + \text{Ent}(w) + \sum_{j=1}^n \ln(\check{w}_j) w_j$ is lower semicontinuous and convex on $\text{cl } \tilde{C}$ and, in particular, the lower contour sets of this function are closed and convex subsets of $\text{cl } \tilde{C}$, that is, they are compact subsets. At the same time, since $d(w) = \infty$ for all $w \in \text{cl } \tilde{C} \setminus \tilde{C}$, they are subsets of $\tilde{C}$, proving that $w \mapsto d(w) + \text{Ent}(w) + \sum_{j=1}^n \ln(\check{w}_j) w_j$ is lower coercive and quasiconvex on $\tilde{C}$ and so is $w \mapsto \mathfrak{G}(w, Q)$. By Lemma 18 and since $G^* : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ is such that

$$G^*(Q) = \min_{w \in \tilde{C}} \mathfrak{G}(w, Q) \quad \forall Q \in \mathbb{R}_+^n,$$

then

$$K_{G^*}(P) = \max_{w \in \tilde{C}} K_w(P) = \max_{w \in \tilde{C}} \frac{\prod_{j=1}^n P_j^{w_j}}{e^{d(w)}} \quad \forall P \in \mathbb{R}_{++}^n.$$

Since $\kappa(p) = \kappa_{G^*}(p) = \ln(K_{G^*}(\exp(p)))$ for all $p \in \mathbb{R}^n$, it follows that

$$\kappa(p) = \ln(K_{G^*}(\exp(p))) = \ln \left(\max_{w \in \tilde{C}} \frac{\prod_{j=1}^n e^{w_j p_j}}{e^{d(w)}} \right) = \max_{w \in \tilde{C}} \left\{ \sum_{j=1}^n w_j p_j - d(w) \right\} \quad \forall p \in \mathbb{R}^n,$$

proving the convexity of κ . □

F.5 Smoothness and density

In this section, we show that smooth unit log-cost functions are dense. In light of Theorem 4, we can therefore think that for any $G \in \mathcal{G}_{\text{nor}}$ there exists a sequence $\{G_m\}_{m \in \mathbb{N}} \subseteq \mathcal{G}_{\text{nor}}$ such that $\kappa_{G_m} \in C^\infty(\mathbb{R}^n)$ for all $m \in \mathbb{N}$ and $\lim_m \kappa_{G_m}(p) = \kappa_G(p)$ for all $p \in \mathbb{R}^n$. By Arzela-Ascoli's Theorem and since the sequence $\{\kappa_{G_m}\}_{m \in \mathbb{N}}$ is uniformly bounded over compacta and Lipschitz

of order 1, this implies that $\kappa_{G_m} \xrightarrow{d} \kappa_G$ where d is the distance on $C(\mathbb{R}^n)$ metrizing convergence over compacta. This is our density result (cf. Proposition 13).

As one might suspect, we carry out these approximations by convolution with mollifiers. To this end, we need an intermediate approach. This intermediate approach brings us to consider the following new objects. Given a function $K \in \mathcal{K}_{\text{nor}}$, we define $\mathfrak{K} : \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$\mathfrak{K}(p) = K(\exp(p)) \quad \forall p \in \mathbb{R}^n$$

where as usual $\exp(p)$ denotes the vector whose i -th component is e^{p_i} . Recall that $\mathbb{R}^n$ is endowed with the supnorm $\|\cdot\|_\infty$ and we denote by B the unit ball. Consider a function $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $\varphi \geq 0$, $\int_{\mathbb{R}^n} \varphi(x) dx = 1$, and $\text{supp } \varphi \subseteq B$. For each $\varepsilon > 0$ define $\varphi_\varepsilon(x) = \varepsilon^{-n} \varphi(\frac{x}{\varepsilon})$ for all $x \in \mathbb{R}^n$. By the Change of Variable Formula, note that $\varphi_\varepsilon \geq 0$, $\int_{\mathbb{R}^n} \varphi_\varepsilon(x) dx = 1$, and $\text{supp } \varphi_\varepsilon \subseteq \varepsilon B$ for all $\varepsilon > 0$. Given a function $K \in \mathcal{K}_{\text{nor}}$ and $\varepsilon > 0$, we define $\mathfrak{K}_\varepsilon : \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$\mathfrak{K}_\varepsilon(\bar{p}) = (\varphi_\varepsilon \star \mathfrak{K})(\bar{p}) = \int_{\mathbb{R}^n} \varphi_\varepsilon(\bar{p} - p) \mathfrak{K}(p) dp \quad \forall \bar{p} \in \mathbb{R}^n.$$

We define $K_\varepsilon : \mathbb{R}_{++}^n \rightarrow \mathbb{R}_{++}$ by $K_\varepsilon(P) = \frac{1}{\mathfrak{K}_\varepsilon(0)} \mathfrak{K}_\varepsilon(\ln P)$ where $\ln P$ is the vector whose i -th component is $\ln P_i$ for all $P \in \mathbb{R}_{++}^n$. As before, we define $\kappa_\varepsilon : \mathbb{R}^n \rightarrow \mathbb{R}$ by $\kappa_\varepsilon(p) = \ln(K_\varepsilon(\exp(p)))$ for all $p \in \mathbb{R}^n$.

Proposition 13. *If $K \in \mathcal{K}_{\text{nor}}$ and $\{\varepsilon_m\}_{m \in \mathbb{N}} \subseteq (0, 1)$ such that $\lim_m \varepsilon_m = 0$, then $K_{\varepsilon_m} \in \mathcal{K}_{\text{nor}}$ and $\kappa_{\varepsilon_m} \in C^\infty(\mathbb{R}^n)$ for all $m \in \mathbb{N}$ and $\kappa_{\varepsilon_m} \xrightarrow{d} \kappa$.*

Proof. Since K is continuous, so is $\mathfrak{K}$ and, in particular, $\mathfrak{K} \in L_{\text{loc}}^1(\mathbb{R}^n)$. By (Brezis, 2011, Propositions 4.20 and 4.21), it follows that $\mathfrak{K}_{\varepsilon_m} \in C^\infty(\mathbb{R}^n)$ for all $m \in \mathbb{N}$ and $\mathfrak{K}_{\varepsilon_m}$ converges uniformly to $\mathfrak{K}$ over compacta. By the Change of Variable Formula, recall that convolution is commutative. It follows that

$$\mathfrak{K}_{\varepsilon_m}(\bar{p}) = (\varphi_{\varepsilon_m} \star \mathfrak{K})(\bar{p}) = (\mathfrak{K} \star \varphi_{\varepsilon_m})(\bar{p}) = \int_{\mathbb{R}^n} \mathfrak{K}(\bar{p} - p) \varphi_{\varepsilon_m}(p) dp \quad \forall \bar{p} \in \mathbb{R}^n, \forall m \in \mathbb{N}.$$

Since $K > 0$ and $\varphi_{\varepsilon_m} \geq 0$ and $\int_{\mathbb{R}^n} \varphi_{\varepsilon_m}(x) dx = 1$ for all $m \in \mathbb{N}$, we have that $\mathfrak{K} > 0$ and, in particular, $\mathfrak{K}_{\varepsilon_m} > 0$ for all $m \in \mathbb{N}$. It follows that K_{ε_m} is well defined and continuous for all $m \in \mathbb{N}$. Moreover, by construction, we have that $K_{\varepsilon_m}(e) = 1$ for all $m \in \mathbb{N}$. Since K is monotone, so is $\mathfrak{K}$. Since $\mathfrak{K}$ is monotone and $\varphi_{\varepsilon_m} \geq 0$ and $\int_{\mathbb{R}^n} \varphi_{\varepsilon_m}(p) dp = 1$ for all $m \in \mathbb{N}$, it follows that

$$\hat{p} \geq \tilde{p} \implies \mathfrak{K}_{\varepsilon_m}(\hat{p}) = \int_{\mathbb{R}^n} \mathfrak{K}(\hat{p} - p) \varphi_{\varepsilon_m}(p) dp \geq \int_{\mathbb{R}^n} \mathfrak{K}(\tilde{p} - p) \varphi_{\varepsilon_m}(p) dp = \mathfrak{K}_{\varepsilon_m}(\tilde{p}) \quad \forall m \in \mathbb{N},$$

proving monotonicity of $\mathfrak{K}_{\varepsilon_m}$ and K_{ε_m} for all $m \in \mathbb{N}$. Since K is positive homogeneous and

$\varphi_{\varepsilon_m} \geq 0$ and $\int_{\mathbb{R}^n} \varphi_{\varepsilon_m}(p) dp = 1$ for all $m \in \mathbb{N}$, it follows that

$$\begin{aligned}\mathfrak{K}_{\varepsilon_m}(\bar{p} + he) &= \int_{\mathbb{R}^n} \mathfrak{K}(\bar{p} + he - p) \varphi_{\varepsilon_m}(p) dp = \int_{\mathbb{R}^n} K(\exp(\bar{p} + he - p)) \varphi_{\varepsilon_m}(p) dp \\ &= e^h \int_{\mathbb{R}^n} K(\exp(\bar{p} - p)) \varphi_{\varepsilon_m}(p) dp = e^h \int_{\mathbb{R}^n} \mathfrak{K}(\bar{p} - p) \varphi_{\varepsilon_m}(p) dp \\ &= e^h \mathfrak{K}_{\varepsilon_m}(\bar{p}) \quad \forall \bar{p} \in \mathbb{R}^n, \forall h \in \mathbb{R}, \forall m \in \mathbb{N}.\end{aligned}$$

It follows that

$$\begin{aligned}K_{\varepsilon_m}(\lambda P) &= \frac{1}{\mathfrak{K}_{\varepsilon_m}(0)} \mathfrak{K}_{\varepsilon_m}(\ln(\lambda P)) = \frac{1}{\mathfrak{K}_{\varepsilon_m}(0)} \mathfrak{K}_{\varepsilon_m}(\ln(P) + \ln(\lambda) e) \\ &= \frac{1}{\mathfrak{K}_{\varepsilon_m}(0)} e^{\ln(\lambda)} \mathfrak{K}_{\varepsilon_m}(\ln(P)) = \lambda K_{\varepsilon_m}(P), \quad \forall P \in \mathbb{R}_{++}^n, \forall \lambda > 0, \forall m \in \mathbb{N},\end{aligned}$$

proving that K_{ε_m} is positive homogeneous for all $m \in \mathbb{N}$. Consider $P, \tilde{P} \in \mathbb{R}_{++}^n$ and $\lambda \in (0, 1)$. Given a vector $p \in \mathbb{R}^n$ define P_p (resp. $\tilde{P}_p$) as the vector whose i -th component is $P_i / \exp(p_i)$ (resp. $\tilde{P}_i / \exp(p_i)$). Since K is concave, we have that

$$\begin{aligned}&\mathfrak{K}_{\varepsilon_m}(\ln(\lambda P + (1 - \lambda) \tilde{P})) \\ &= \int_{\mathbb{R}^n} \mathfrak{K}(\ln(\lambda P + (1 - \lambda) \tilde{P}) - p) \varphi_{\varepsilon_m}(p) dp \\ &= \int_{\mathbb{R}^n} \mathfrak{K}(\ln(\lambda P_p + (1 - \lambda) \tilde{P}_p)) \varphi_{\varepsilon_m}(p) dp \\ &= \int_{\mathbb{R}^n} K(\lambda P_p + (1 - \lambda) \tilde{P}_p) \varphi_{\varepsilon_m}(p) dp \\ &\geq \lambda \int_{\mathbb{R}^n} K(P_p) \varphi_{\varepsilon_m}(p) dp + (1 - \lambda) \int_{\mathbb{R}^n} K(\tilde{P}_p) \varphi_{\varepsilon_m}(p) dp \\ &= \lambda \int_{\mathbb{R}^n} \mathfrak{K}(\ln(P) - p) \varphi_{\varepsilon_m}(p) dp + (1 - \lambda) \int_{\mathbb{R}^n} \mathfrak{K}(\ln(\tilde{P}) - p) \varphi_{\varepsilon_m}(p) dp \\ &= \lambda \mathfrak{K}_{\varepsilon_m}(\ln(P)) + (1 - \lambda) \mathfrak{K}_{\varepsilon_m}(\ln(\tilde{P})) \quad \forall m \in \mathbb{N},\end{aligned}$$

proving concavity of K_{ε_m} for all $m \in \mathbb{N}$. We can conclude that $K_{\varepsilon_m} \in \mathcal{K}_{\text{nor}}$ for all $m \in \mathbb{N}$. Since $\mathfrak{K}_{\varepsilon_m} \in C^\infty(\mathbb{R}^n)$ and $\mathfrak{K}_{\varepsilon_m} > 0$ for all $m \in \mathbb{N}$ and $\ln$ is infinitely many times differentiable on $(0, \infty)$, we can conclude that $\kappa_{\varepsilon_m} \in C^\infty(\mathbb{R}^n)$ for all $m \in \mathbb{N}$. Since $\lim_m \mathfrak{K}_{\varepsilon_m}(p) = \mathfrak{K}(p)$ for all $p \in \mathbb{R}^n$ and $\ln$ is continuous on $(0, \infty)$, we have that $\lim_m \mathfrak{K}_{\varepsilon_m}(0) = \mathfrak{K}(0) = 1$ and, in particular,

$$\begin{aligned}\lim_m \kappa_{\varepsilon_m}(p) &= \lim_m [\ln \mathfrak{K}_{\varepsilon_m}(p) - \ln \mathfrak{K}_{\varepsilon_m}(0)] = \lim_m \ln \mathfrak{K}_{\varepsilon_m}(p) - \lim_m \ln \mathfrak{K}_{\varepsilon_m}(0) \\ &= \ln \mathfrak{K}(p) = \kappa(p) \quad \forall p \in \mathbb{R}^n,\end{aligned}$$

proving that κ_{ε_m} converges pointwise to κ . By Arzela-Ascoli's Theorem and since the sequence $\{\kappa_{\varepsilon_m}\}_{m \in \mathbb{N}}$ is uniformly bounded over compacta and Lipschitz of order 1, this implies that $\kappa_{\varepsilon_m} \xrightarrow{d} \kappa$. $\square$

F.6 Semiconcavity

In this section, we prove that if $K \in \mathcal{K}_{\text{nor}}$ and κ is twice continuously differentiable, then $\kappa - \|\cdot\|_2^2/2$ is a concave function where $\|\cdot\|_2$ is the Euclidean norm over $\mathbb{R}^n$. In this section, given two symmetric $n \times n$ matrices A and B , we write $A \succeq B$ if and only if $\langle Ax, x \rangle \geq \langle Bx, x \rangle$ for all $x \in \mathbb{R}^n$. With a small abuse of notation, we denote by I both the $n \times n$ identity matrix and the identity operator.

Lemma 22. *If $K \in \mathcal{K}_{\text{nor}}$ and κ is twice continuously differentiable, then*

$$\max_{h \in B_\infty} \langle \nabla^2 \kappa(p) h, h \rangle \leq 1$$

and, in particular, $I \succeq \nabla^2 \kappa(p)$ for all $p \in \mathbb{R}^n$. Moreover, $\kappa - \|\cdot\|_2^2/2$ is concave.

Proof. Fix $p \in \mathbb{R}^n$ and $w \in \sigma_\kappa(p)$. By Corollary 4 and since $M(w, \cdot)$ and κ are twice continuously differentiable, we have that $\nabla \kappa(p) = \nabla M(w, p)$,

$$\kappa(p+h) = \kappa(p) + \langle \nabla \kappa(p), h \rangle + \frac{1}{2} \langle \nabla^2 \kappa(p) h, h \rangle + R_\kappa(h) \quad \forall h \in \mathbb{R}^n,$$

and

$$M(w, p+h) = M(w, p) + \langle \nabla M(w, p), h \rangle + \frac{1}{2} \langle \nabla^2 M(w, p) h, h \rangle + R_w(h) \quad \forall h \in \mathbb{R}^n$$

where $R_w(\cdot) = o(\|\cdot\|_2^2)$ and $R_\kappa(\cdot) = o(\|\cdot\|_2^2)$. Fix $h \in \mathbb{R}^n$. It follows that

$$\begin{aligned} & \langle \nabla \kappa(p), th \rangle + \frac{1}{2} \langle \nabla^2 \kappa(p) th, th \rangle + R_\kappa(th) \\ &= \kappa(p+th) - \kappa(p) \\ &\leq M(w, p+th) - M(w, p) \\ &= \langle \nabla M(w, p), th \rangle + \frac{1}{2} \langle \nabla^2 M(w, p) th, th \rangle + R_w(th) \quad \forall t > 0, \end{aligned}$$

proving that

$$\frac{1}{2} \langle \nabla^2 \kappa(p) h, h \rangle + \frac{R_\kappa(th)}{t^2} \leq \frac{1}{2} \langle \nabla^2 M(w, p) h, h \rangle + \frac{R_w(th)}{t^2} \quad \forall t > 0.$$

By passing to the limit and since h was arbitrarily chosen, we have that $\langle \nabla^2 \kappa(p) h, h \rangle \leq \langle \nabla^2 M(w, p) h, h \rangle$ for all $h \in \mathbb{R}^n$ and, in particular, $\nabla^2 M(w, p) \succeq \nabla^2 \kappa(p)$. Denote by S_2 the unit sphere induced by $\|\cdot\|_2$ and by B_∞ the unit ball induced by $\|\cdot\|_\infty$. Since $M(w, \cdot)$ is convex, we have that $h \mapsto \langle \nabla^2 M(w, p) h, h \rangle$ is a convex function on $\mathbb{R}^n$. Since $\|\cdot\|_\infty \leq \|\cdot\|_2$, this implies that

$$\begin{aligned} \max_{h \in S_2} \langle \nabla^2 \kappa(p) h, h \rangle &\leq \max_{h \in B_\infty} \langle \nabla^2 \kappa(p) h, h \rangle \leq \max_{h \in B_\infty} \langle \nabla^2 M(w, p) h, h \rangle \\ &= \max_{h \in \text{ext}(B_\infty)} \langle \nabla^2 M(w, p) h, h \rangle \leq 1, \end{aligned}$$

proving that $\langle \nabla^2 \kappa(p) h, h \rangle \leq \|h\|_2^2 = \langle Ih, h \rangle$ for all $h \in \mathbb{R}^n$, that is, $I \succeq \nabla^2 \kappa(p)$. Since p was arbitrarily chosen, the statement follows. $\square$

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